Inverse problem of determining the fuzzy set operations based on the theory of falling shadows

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Abstract: This paper presents the criterions for T-norm and S-nonorm to be defined the intersection and union of fuzzy sets based on the theory of falling shadows.

1. Introduction

Nowadays, the many triangular norms T and triangular conorms S are used in defining fuzzy set operations — union and intersection. Since there are so many T-norms and T-conorms, then to the real applications in which the fuzzy operations union and intersection concern, we may face the problems that how to choose the T-norms and T-conorms.

The theory of falling shadows [8 - 15]presents an approach of determining the fuzzy set opereations according to the correlations of the fuzzy sets.According to the the theory of falling shadows, if the fuzzy sets A and B on the universal set U have the correlations of (i)perfect possitive,(ii)perfect negotive, (iii)independence, then the fuzzy set operations shall respectively be, for every $u \in U$,

- $A \cup B(u) = \max(A(u), B(u))$
  $A \cap B(u) = \min(A(u), B(u))$
- $A \cup B(u) = \min(A(u)+B(u), 1)$
  $A \cap B(u) = \max(A(u)+B(u)-1, 0)$
- $A \cup B(u) = A(u) + B(u) - A(u)B(u)$
  $A \cap B(u) = A(u)B(u)$

which respond repectively the following t-norms and t-conorms

- $S(x, y) = \max(x, y)$
  $T(x, y) = \min(x, y)$
- $S(x, y) = \min(x+y, 1)$
  $T(x, y) = \max(x+y-1, 0)$
- $S(x, y) = x + y - xy$
  $T(x, y) = xy$

Above results tell us, based on the theory of falling shadows, when we use above three groups of T-conorms and T-norms,the correlations of the fuzzy sets A and B should be perfect positive, perfect negotive and independence.

The following problems is opened very naturely: a) given some T-conorms. S and T-norm T, can they be used to define the fuzzy set operations baesd on the theory of falling shadows? and b) if the T-norm and T-conorm can be used to define the fuzzy set operations, what are thecorrelations of fuzzy sets sould be satisfiesd?

These two problems are all the inverse problems of determining fuzzy set operations based on the theory of falling shadows. In this paper, we shall discuss these inverse problems comprehensively. Precisely speaking, in Section 2, we outline the the theory of falling shadows. In section 3, we obtain (*) the condition that T -norm T can be used to define fuzzy intersection is that T is a two-dimentional probabilistic distribution on $[0, 1]\times[0, 1]$, in this time, the correlation of fuzzy sets A and B is T; (**) the condition that a T-conorm S can be used to define fuzzy operation union is that $x + y - S(x, y)$ is two-dimentional probabilistic distribution on $[0, 1]\times[0, 1]$, in this time, the correlation of fuzzy sets A and B is $x + y - S(x, y)$. Also we point out that the dual T-norms and T-conorms in the common sense are not dual according to the theory of falling shadows.

2 Outline of the theory of falling theory

2.1 General theory

Here we give the outline of the theory of falling shadows [8 - 15] which is used in the section 3. For more detail discussions, the readers can refer the mentioned References [8, 15].

Let U be the universal set. For every $u \in U$, let

$$\dot{u} := \{ A;\ u \in A \text{ and } A \in U\}.$$

and $\dot{A} := \{\dot{u}; u \in A\}$

An ordered pair $(\mathcal{P}(U), \mathcal{B})$ is said to be an hyper -measurable structure on U if $\mathcal{B}$ is a sigma-field in $\mathcal{P}(U)$ and satisfies

$$\dot{U} \subset \mathcal{B}$$

Given a probability space $(\Omega, \mathcal{A}, P)$ and an hyper-measurable structure $(\mathcal{P}(U), \mathcal{B})$ on U, a random set on U is defined to be a mapping

$$\xi: \Omega \to \mathcal{P}(U)$$

that is $\mathcal{A}$-$\mathcal{B}$ measurable, i. e., (for every $C \in \mathcal{B}$),

$$\xi^{-1}(C) = \{\omega; \omega \in \Omega \text{ and } \xi(\omega) \in C\} \in \mathcal{A}.$$

suppose that ξ is a random set on U, then the covering function of ξ, $\underline{\xi}$: $U \rightarrow [0, 1]$ is defined by, for every $u \in U$,

$$\underline{\xi}(u) := P(\omega;\ u \in \xi(\omega))$$

Since $\underline{\xi} \in [0, 1]^U$, it represents a fuzzy set A in U and we write

$$\underline{\xi} = A.$$

The random set ξ is called a cloud of A, and A is called the falling shadows of ξ.

Like [8, 9] etc, we assume that when we consider fuzzy sets and do not concern the fuzzy operations, the probability space $(\Omega, \mathcal{A}, P)$ is $([0, 1], \mathcal{B}, m)$, where $\mathcal{B}$ is the Borel-field on [0,1]and m is the *Lebesgue measure.*

For the probability space $([0, 1], \mathcal{B}, m)$, the simple random set whose falling shadow is the fuzzy set A is ξ defined by

$$\xi:\ \lambda \rightarrow A_\lambda, \quad \text{for every } \lambda \in [0, 1]$$

where $A_\lambda = \{u \in U, A(u) \geqslant \lambda\}$. ξ is called the cut-cloud of A.

2.2 The definition of union and intersection of fuzzy sets in the theory of falling shadows

Let A and B be two fuzzy sets in U. Suppose we have chosen the cut-cloud ξ for A with respect to the probability space $(\Omega_1, \mathcal{A}_1, P_1)$ and the cut-cloud η set for B with respect to the probability space $(\Omega_2, \mathcal{A}_2, P_2)$.Where $\Omega_1 = \Omega_2 = [0,1]$, $\mathcal{A}_1 = \mathcal{A}_1 = \mathcal{B}$ ($\mathcal{B}$ is the Broel-field on [0, 1]), $P_1 = P_2$ = the Lebesgue measure m. Then we can consider the jiont probability space $([0, 1]^2, \mathcal{B}^2, P)$ in which both of the projections of P on [0, 1] are m. Now we define the union and intersection of A and B as the fallings of $\xi \cup \eta$ and $\xi \cap \eta$ respectively, where $\xi \cup \eta$ and $\xi \cap \eta$ are two random sets from $[0, 1] \times [0, 1]$ to $\mathcal{P}(U)$:

$$\xi \cup \eta(\lambda\ \mu) = A_\lambda \cup B_\mu$$

and

$$\xi \cap \eta\ (\lambda\ \mu) = A_\lambda \cap B_\mu$$

for $(\lambda\ \mu) \in [0, 1] \times [0, 1]$.

Let the probability P on $\mathcal{B}^2$ be given and be fixed. Then the union and intersection of A abd B are , for every $u \in U$,

$$(A \cup B)(u) = P((\xi \cup \eta)^{-1}(\dot{u})) = P(\{(\lambda\ \mu);\ u \in A_\lambda \cup B_\mu\})$$

$$(A \cap B)(u) = P((\xi \cap \eta)^{-1}(\dot{u})) = P(\{(\lambda\ \mu);\ u \in A_\lambda \cap B_\mu\}).$$

Tan, Wang and Lee [8] consider the probability P's with determined by the following probabilistic distributions p's on $[0, 1] \times [0, 1]$:

(1) P is uniformly distributed on the diagonal ($\lambda=\mu$) of $[0, 1]\times[0, 1]$, in this case,

$$(A\cup B)(u) = \max(A(u), B(u))$$
$$(A\cap B)(u) = \min(A(u), B(u))$$

(2) P is uniformly distributed on the anti-diagonal ($\lambda+\mu=1$) of $[0, 1]\times[0, 1]$, in this case,

$$(A\cup B)(u) = \max(A(u)+ B(u), 1)$$
$$(A\cap B)(u) = \min((A(u)+ B(u))-1, 0)$$

(3) P is uniformly distributed on $[0, 1]^2$, in this case

$$(A\cup B)(u) = A(u)+ B(u) -A(u)B(u)$$
$$(A\cap B)(u) = A(u)B(u)$$

3 Criterions

3.1 T-norm (triangular norm) and T-conorm (triangular conorm)

Definition 3.1.1. A function T from $[0,1]\times[0, 1]\rightarrow [0, 1]$ is said a T -norm (also called triangular norm) if T satisfies

(T1) $T(1, a) = a$, for every $a\in[0, 1]$;

(T2) $T(a, b) = T(b, a)$, for every $a, b\in[0, 1]$;

(T3) $T(a, b) \leqslant T(c, d)$, for every a, b , c and $d\in[0,1]$ such that $a\leqslant c$, $b\leqslant d$;

(T4) $T(T(a, b), c) = T(a, T(b, c))$, for every $a, b, c\in[0, 1]$.

Definition 3.1.2 A function S from $[0,1]\times[0, 1]\rightarrow [0, 1]$ is said a S -norm (also called triangular conorm) if S satisfies

(S1) $S(0, a) = a$, for every $a\in[0, 1]$;

(S2) $S(a, b) = S(b, a)$, for every $a, b\in[0, 1]$;

(S3) $S(a, b)\leqslant S(c, d)$, for every a, b, c and $d\in[0,1]$ such that $a\leqslant c$, $b\leqslant d$;

(S4) $S(S(a, b), c) = S(a, S(b, c))$, for every $a, b, c\in[0, 1]$.

In this paper, we always denote the T-norm and T-conorm by T and S resctively.

Definition 3.1.3 T-norm T and T-conorm S are called dual, if for every $a, b\in[0, 1]$,

$$T(a, b) + S(1- a, 1-b) = 1.$$

Definition 3.1.4 For T-norm T and S-norm S which are dual in the sense of Definition 3.1.3, the union and intersection of fuzzy sets A and B are defined by

$$(A\cup B)(u) = S(A(u), B(u)),$$
$$(A\cap B)(u) = T(A(u), B(u)).$$

3.2 Criterions to the T-norm

Here, we consider that in what condition that the intersection of fuzzy sets A and B defined by a T-norm T can be dertermined by a probabilistic distribution P on $[0,1]\times[0, 1]$,according to the theory of falling shadows.

Suppose that the intersection of fuzzy sets A and B is defined by a T -norm T, i.e.,

$$(A\cap B)(u) = T(A(u), B(u)).$$

and also, according to the theory of falling shadows, determined by the probabilistic distributon P on $[0, 1]\times[0, 1]$, i.e.,

$$\begin{aligned}(A\cap B)(u) &= P((\xi\cap\eta)^{-1}(u)) \\ &= P(\{(\lambda\ \mu);\ u\in A_\lambda\cap B_\mu\}). \\ &= P(\{(\lambda\ \mu);\ \lambda\leqslant A(u) \text{ and } \mu\leqslant B(u)\}),\end{aligned}$$

we have

$$P(\{(\lambda\ \mu);\ \lambda\leqslant A(u) \text{ and } \mu\leqslant B(u)\}) = T(A(u), B(u)).$$

Note the distribution function of $(\lambda\ \mu)$ by F(a, b) , that is

$$F(a, b) = \int\int_{[0,\ a]\times[0,\ b]} P(dxdy),\ a,\ b\in[0, 1],$$

Then

$$F(A(u), B(u)) = T(A(u), B(u)), \text{ for every } u\in U.$$

When we consider the general fuzzy sets A and B , not the special ones mentioned above, we should assume that for every a, b in [0, 1],

$$T(a, b) = F(a, b)$$

that is to say that T is a probabilistic distribution on $[0, 1]\times[0, 1]$. In this time, T's two projections of λ and μ are all uniform distributions on [0, 1].

Basing on the above discussions, we present the following:

Criteria (i) The condition that A T-norm T can be used to define intersection of fuzzy sets according to the theory of falling shadows is that T is a probabilistic distribution function on $[0, 1]\times[0, 1]$, and in this time, T is also the probabilistic distribution function of $(\lambda\ \mu)$.

Since T-norm T satisfies (T2) and (T3), then the above Criterion can be also described by

Criterion (ii) The condition that A T-norm T can be used to define intersection of fuzzy sets according to the theory of falling shadows is that T satisfies that for every a_1, b_1, a_2 and b_2 in [0,1] such that $a_1\leqslant a_2$, $b_1\leqslant b_2$, there holds

$$F(a_1,b_1) + F(a_2,b_2) - F(a_1, b_2) - F(a_2, b_1) \geqslant 0$$

and in this time, T is also the probabilistic distribution function of $(\lambda\ \mu)$.

If the T-norm T has the continuous two-order partial derivative $\partial^2 T/\partial a\partial b$, there hohds

$$\int\int_{[0,1]\times[0,1]} (\partial^2 T/\partial a\partial b)\,da\,db$$
$$= \int_{[0,1]} [\partial T(a, b)/\partial a]_{b=0}^{b=1} da$$
$$= \int_{[0,1]} [1 - 0] da = 1,$$
and hence, we have

Criterion (iii) If the T-norm T has the continuous two-order partial derivative $\partial^2 T/\partial a\partial b$, Then The condition that A T-norm T can be used to define intersection of fuzzy sets according to the theory of falling shadows is that for every a, $b \in [0, 1]$,

$$\partial^2 T/\partial a\partial b(a, b) \geqslant 0$$

and in this time, the probabilistic distribution density of $(\lambda\ \mu)$ is $\partial^2 T/\partial a\partial b$.

3.3 Criterion to T-conorm

Now we turn to the problem: in what condition the union of fuzzy sets defined by S-norm S can be determined by a probabilistic distribution P on $[0, 1]\times[0, 1]$.

Suppose that the union of fuzzy sets A and B is defined by a T-conorm S , i.e.,

$$(A\cup B)(u) = S(A(u), B(u)),$$

and also, according to the theory of falling shadows, determined by the probabilistic distributon p on $[0, 1]\times[0, 1]$, i.e.,

$$(A\cup B)(u) = P((\xi \cup \eta)^{-1}(\dot{u}))$$
$$= P(\{(\lambda\ \mu);\ u\in A_\lambda \cup B_\mu\}).$$
$$= P(\{(\lambda\ \mu);\ \lambda\leqslant A(u) \text{ or } \mu\leqslant B(u)\}),$$

we have

$$P(\{(\lambda\ \mu);\ \lambda\leqslant A(u) \text{ or } \mu\leqslant B(u)\}),$$
$$= S(A(u), B(u)).$$

Note the distribution function of $(\lambda\ \mu)$ by F(a, b) , that is

$$F(a, b) = \int\int_{[0,a]\times[0,b]} P(dxdy),\ a,\ b\in[0, 1],$$

Then for every $u\in U$

$$F(A(u),1) + F(1, B(u) - F(A(u), B(u)) = S(A(u), B(u)).$$

when we consider the general fuzzy sets A and B , not the special ones mentioned above, we should assume that for every a, b in [0, 1],

$$S(a, b) = F(a, 1) + F(1, b) - F(a, b) = a + b - F(a,b)$$

since F has uniform projection.

Basing on the above discussions, we present the following:
Criteria (iv) A T-conorm S can be to a probabilistic distribution P on $[0, 1]\times[0, 1]$ according to the theory of falling shadows is that a + b - S(a, b) is a probabilistic distribution function on

$[0, 1]\times[0, 1]$, and in this time, $a + b - S(a, b)$ be also the probabilistic distribution of $(\lambda\ \mu)$.

Also, we can easyly present

Criterion (v) The condition that A T-conorm S can beusedto define union of fuzzy sets according to the theory of falling shadows is that Ssatisfies that for every a_1, b_1, a_2 and b_2 in [0,1] such that $a_1 \leqslant a_2$, $b_1 \leqslant b_2$, there holds

$$S(a_1,b_1) + S(a_2,b_2) - S(a_1, b_2) - S(a_2, b_1) \leqslant 0$$

and in this time, $a + b - S(a, b)$ is also the probabilistic distribution function of $(\lambda\ \mu)$.

When S has continuous two-order partial devrivative $\partial^2 S/\partial a\partial b$, there hohds

$$\int\int_{[0,1]\times[0,1]} (\partial^2 S/\partial a\partial b)\, da db = -1,$$

and hence, we have

Criterion (vi) If the T-conorm S has the continuous two-order partial derivative $\partial^2 S/\partial a\partial b$, Then the condition that a T-conorm S can be used to define union of fuzzysets according to the theory of falling shadows is that for every $a, b \in [0, 1]$,

$$(\partial^2 S/\partial a\partial b)(a, b) \leqslant 0$$

and in this time, S is also the probabilistic distribution density of $(\lambda\ \mu)$.is $-\partial^2 S/\partial a\partial b$.

3.4 On the duality

According the the Definition 3.1.1 -3.1.3, we have T-norm and T-conorm exist dually. However, in the sense of theory of falling shadows, if a T-norm T and T-conorm S are related to the same distribution P in defining the intersection and union of fuzzy sets, they must satisfy that

$$T(a, b) = a + b - S(a, b)$$

That is

(#) $T(a, b) + S(a, b) = a + b.$

Sometimes,the dual T-norm T and T-conorm S satisfy (#),like the cases mentioned in section 1. In this time, we can only consider one of T -norm and T-conorm.

In the general case, the dual T-norm T and and S-norm S in the sense of Definition 3.1.1 - 3.1.3 do not satisfy (%), and hence, we should study the T-norm T and T-conorm S respectively.

In addition, from Criterion (ii) and and (v), we can show that if T -norm T and T-conorm S are dual in the sense of Definition 3.1.1 -3.1.3, that the intersection of fuzzy sets A and B defined by a T-norm T can be dertermined by a probabilistic distribution P on [0, 1]×[0, 1],is equipvalent to which the union of fuzzy sets A and B defined by a T-conorm S can be dertermined by aprobabilistic distribution P on [0, 1]×[0, 1], according to the theory of falling shadows, but, if (%) is not true, the the P's are not the same one.

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