LOWER CUT SET, DECOMPOSITION THEOREM AND REPRESENTATION THEOREM

CHEN TUYUN

Department of Mathematics, Liaoning Normal University,
Dalian 116022, *P. R. China*

ABSTRACT

This paper gives concepts of lower cut sets of fuzzy sets and order sets embeddings, we discuss some properties of lower cut sets and give new decomposition theorem and representation theorem of fuzzy sets.

keywords: fuzzy sets, lower cut sets, strong lower cut set, order set embeddings, decomposition theorem, representation theorem.

1. Lower cut sets and their properties

Let $\mathscr{F}(A)=\{A \mid A: X\to[0,1]$ is a mapping$\}$. For $A\in \mathscr{F}(X)$, $\lambda\in[0,1]$, $A_\lambda=\{x\mid x\in X, A(x)\geqslant\lambda\}$, $A_{\underset{\cdot}{\lambda}}=\{x\mid x\in X, A(x)>\lambda\}$ are called as λ-cut set and λ-strong cut set of fuzzy set A respectively. In this paper, we call A_λ, $A_{\underset{\cdot}{\lambda}}$ as λ-upper cut set and λ-strong upper cut set of fuzzy set A respeceively. Now, we give two new concepts as following.

Definition 1 Let $A\in \mathscr{F}(X)$, $\lambda\in[0,1]$ and

$$A^\lambda=\{x\mid x\in X, A(x)\leqslant\lambda\},\ A^{\underset{\cdot}{\lambda}}=\{x\mid x\in X, A(x)<\lambda\},$$

A^λ, $A^{\underset{\cdot}{\lambda}}$ are called as λ-lower cut set and λ-strong lower cut set of fuzzy set A respectively.

Clearly, we have the following conclusions.

Proposition 1 (1) $\lambda_1<\lambda_2\Rightarrow A^{\lambda_1}\subseteq A^{\lambda_2}$, $A^{\underset{\cdot}{\lambda_1}}\subseteq A^{\underset{\cdot}{\lambda_2}}$; (2) $A^\lambda\supseteq A^{\underset{\cdot}{\lambda}}$; (3) $\lambda>\mu\Rightarrow A^{\underset{\cdot}{\lambda}}\supseteq A^\mu$.

Proposition 2 (1) $(A\cap B)^\lambda=A^\lambda\cup B^\lambda$, $(A\cup B)^\lambda=A^\lambda\cap B^\lambda$. In general $(\bigcup_{t\in T}A_t)^\lambda=\bigcap_{t\in T}A_t^\lambda$, $(\bigcap_{t\in T}A_t)^\lambda\supseteq\bigcup_{t\in T}A_t^\lambda$.

(2) $(A\cap B)^{\underset{\cdot}{\lambda}}=A^{\underset{\cdot}{\lambda}}\cup B^{\underset{\cdot}{\lambda}}$, $(A\cup B)^{\underset{\cdot}{\lambda}}=A^{\underset{\cdot}{\lambda}}\cap B^{\underset{\cdot}{\lambda}}$. In general $(\bigcap_{t\in T}A_t)^{\underset{\cdot}{\lambda}}=\bigcup_{t\in T}A_t^{\underset{\cdot}{\lambda}}$, $(\bigcup_{t\in T}A_t)^{\underset{\cdot}{\lambda}}\subseteq\bigcap_{t\in T}A_t^{\underset{\cdot}{\lambda}}$.

Proposition 3 (1) $(A^\lambda)^c=(A^c)^{\underset{\cdot}{1-\lambda}}$, (2) $(A^{\underset{\cdot}{\lambda}})^c=(A^c)^{1-\lambda}$.

Proposition 4 (1) $A^{(\bigvee_{t\in T}\alpha_t)}=\bigcup_{t\in T}A^{\alpha_t}$, (2) $A^{(\bigwedge_{t\in T}\alpha_t)}=\bigcap_{t\in T}A^{\alpha_t}$, where $\bigvee_{t\in T}\alpha_t=\sup\{\alpha_t\mid t\in T\}$, $\bigwedge_{t\in T}\alpha_t=\inf\{\alpha_t\mid t\in T\}$.

2. Decomposition Theorem

Let C be a subset of set $X, \lambda \in [0,1]$. We define λC as a fuzzy subset of X and

$$(\lambda C)(x)=\begin{cases}\lambda, & \text{if } x \in C; \\ 1, & \text{if } x \notin C.\end{cases}$$

Let A be a fuzzy subset of X, then we have

Theorem 1 $A=\bigcap_{\lambda \in [0,1]} \lambda A^{\lambda}$, i. e. , $A(x)=\inf\{\lambda \mid \lambda \in [0,1], x \in A^{\lambda}\}=\inf\{\lambda \mid \lambda \in [0,1], A(x) \leqslant \lambda\}$.

Theorem 2 $A=\bigcap_{\lambda \in [0,1]} \lambda A^{\underset{\cdot}{\lambda}}$, i. e. , $A(x)=\inf\{\lambda \mid \lambda \in [0,1], x \in A^{\underset{\cdot}{\lambda}}\}=\inf\{\lambda \mid \lambda \in [0,1], A(x)<\lambda\}$.

Theorem 3 Let $H:[0,1] \to \mathscr{P}(X), \lambda \mapsto H(\lambda)$ satisfies: $A^{\underset{\cdot}{\lambda}} \subseteq H(\lambda) \subseteq A^{\lambda}$, then

$(1)\ \lambda_1<\lambda_2 \Rightarrow H(\lambda_1) \subseteq H(\lambda_2)$;

$(2)\ A=\bigcap_{\lambda \in [0,1]} \lambda H(\lambda)$, i. e. , $A(x)=\inf\{\lambda \mid \lambda \in [0,1], x \in H(\lambda)\}$;

$(3)\ A^{\lambda}=\bigcap_{\alpha>\lambda} H(\alpha), A^{\underset{\cdot}{\lambda}}=\bigcup_{\alpha<\lambda} H(\alpha)$.

Proof (1) $\lambda_1<\lambda_2 \Rightarrow H(\lambda_2) \supseteq A^{\underset{\cdot}{\lambda_2}} \supseteq A^{\lambda_1} \supseteq H(\lambda_1)$.

$(2)\ A^{\underset{\cdot}{\lambda}} \subseteq H(\lambda) \subseteq A^{\lambda} \quad \Rightarrow \lambda A^{\underset{\cdot}{\lambda}} \supseteq \lambda H(\lambda) \supseteq \lambda A^{\lambda} \quad \Rightarrow \bigcap_{\lambda \in [0,1]} \lambda A^{\underset{\cdot}{\lambda}} \supseteq \bigcap_{\lambda \in [0,1]} \lambda H(\lambda) \supseteq \bigcap_{\lambda \in [0,1]} \lambda A^{\lambda}$ $\Rightarrow A=\bigcap_{\lambda \in [0,1]} \lambda H(\lambda)$.

$(3)\ \alpha>\lambda \Rightarrow H(a) \supseteq A^{\underset{\cdot}{\alpha}} \supseteq A^{\lambda} \Rightarrow \bigcap_{\alpha>\lambda} H(\alpha) \supseteq A^{\lambda}$. Since $\bigcap_{\alpha>\lambda} H(\alpha) \subseteq \bigcap_{\alpha>\lambda} A^{\alpha}=A^{(\bigwedge_{\alpha>\lambda} \alpha)}=A^{\lambda}$, so $A^{\lambda}=\bigcap_{\alpha>\lambda} H(\alpha)$.

Similarly, $A^{\underset{\cdot}{\lambda}}=\bigcup_{\alpha<\lambda} H(\alpha)$.

3. Order set embeddings and representation theorem

Definition 2 Let $\mathscr{P}(X)$ be power set of set X, $H:[0,1] \to \mathscr{P}(X), \lambda \mapsto H(\lambda)$ be a mapping and satisfies $\lambda<\mu \Rightarrow H(\lambda) \subseteq H(\mu)$. We call H as a order set embedding over X. $\mathscr{U}(X)$ is denoted as a set of all order sets embedding over X.

Definition 3 In $\mathscr{U}(X)$, we define operations $\cup, \cap, \subset$ as following.

$$H_1 \cup H_2:(H_1 \cup H_2)(\lambda)=H_1(\lambda) \cap H_2(\lambda);$$
$$H_1 \cap H_2:(H_1 \cap H_2)(\lambda)=H_1(\lambda) \cup H_2(\lambda);$$
$$\bigcup_{r \in \Gamma} H_r:(\bigcup_{r \in \Gamma} H_r)(\lambda)=\bigcap_{r \in \Gamma} H_r(\lambda);$$
$$\bigcap_{r \in \Gamma} H_r:(\bigcap_{r \in \Gamma} H_r)(\lambda)=\bigcup_{r \in \Gamma} H_r(\lambda);$$
$$H^c:H^c(\lambda)=(H(1-\lambda)^c)=(H(1-\lambda))^c.$$

Theorem 4 Let $T:\ \mathscr{U}(X) \to \mathscr{F}(X),\ H \mapsto T(H)=\bigcap_{\lambda \in [0,1]} \lambda H(\lambda)$, where $(\bigcap_{\lambda \in [0,1]} \lambda H(\lambda))(x)=\inf\{\lambda \mid \lambda \in [0,1], x \in H(\lambda)\}$, then T is a homomorphism from

$(\mathscr{U}(X), \cup, \cap, \subset)$ to $(\mathscr{F}(X), \cup, \cap, \subset)$, i.e.,

(1) for any $A \in \mathscr{F}(X)$ there is a $H \in \mathscr{U}(X)$, such that $T(H)=A$;

(2) $T(\bigcup_{r\in\Gamma} H_r)=\bigcup_{r\in\Gamma} T(H_r), \forall\ H_r \in \mathscr{U}(X)$;

(3) $T(\bigcap_{r\in\Gamma} H_r)=\bigcap_{r\in\Gamma} T(H_r), \forall\ H_r \in \mathscr{U}(X)$;

(4) $T(H^c)=(T(H))^c$,

and T satisfies

(5) $T(H)_{\lambda} \subseteq H(\lambda) \subseteq T(H)^{\lambda}, \forall\ \lambda \in [0,1]$;

(6) $T(H)^{\lambda}=\bigcap_{\alpha>\lambda} H(\alpha), (\lambda \neq 1), T(H)_{\lambda}=\bigcup_{\alpha<\lambda} H(\alpha), (\lambda \neq 0)$.

Proof (1) For any $A \in \mathscr{F}(X)$, let $H \in \mathscr{U}(X)$ and $H(\lambda)=A^{\lambda}$. By theorem 1, $T(H)=A$.

We prove (5) firstly. Let $x \in H(\lambda)$, then $T(H)(x)=\inf\{\alpha \mid \alpha \in [0,1], x \in H(\alpha)\} \leqslant \lambda$, it follows that $X \in T(H)^{\lambda}$ and consequently $H(\lambda) \subseteq T(H)^{\lambda}$.

Assume that $X \notin H(\lambda)$, then $x \notin H(\alpha)$ for any $\alpha \leqslant \lambda$, it follows that if $x \in H(\alpha)$, then $\alpha > \lambda$. Hence $T(H)(x)=\inf\{\alpha \mid x \in H(\alpha)\} \geqslant \lambda$ and $x \notin T(H)_{\lambda}$. It foolws that $T(H)_{\lambda} \subseteq H(\lambda)$, then $T(H)_{\lambda} \subseteq H(\lambda) \subseteq T(H)^{\lambda}$ for any $\lambda \in [0,1]$.

By theorem 3, we can obtain $T(H)^{\lambda}=\bigcap_{\alpha>\lambda} H(\alpha), T(H)_{\lambda}=\bigcup_{\alpha>\lambda} H(\alpha)$.

Now, we prove (2) (3) (4) of theorem 4.

(2) $\forall\ \lambda \in [0,1], T(\bigcup_{r\in\Gamma} H_r)^{\lambda}=\bigcap_{\alpha>\lambda}(\bigcup_{r\in\Gamma} H_r)(\alpha)=\bigcap_{\alpha>\lambda}\bigcap_{r\in\Gamma} H_r(\alpha)=\bigcap_{r\in\Gamma}\bigcap_{\alpha>\lambda} H_r(\alpha)=\bigcap_{r\in\Gamma} T(H_r)^{\lambda}=(\bigcup_{r\in\Gamma} T(H_r))^{\lambda}$. By theorem 1, we have $T(\bigcup_{r\in\Gamma} H_r)=\bigcup_{r\in\Gamma} T(H_r)$.

(3) $\forall\ \lambda \in [0,1], T(\bigcap_{r\in\Gamma} H_r)_{\lambda}=\bigcup_{\alpha<\lambda}(\bigcap_{r\in\Gamma} H_r)(\alpha)=\bigcup_{\alpha<\lambda}\bigcup_{r\in\Gamma} H_r(\alpha)=\bigcup_{r\in\Gamma}(\bigcup_{\alpha<\lambda} H_r(\alpha))=\bigcup_{r\in\Gamma} T(H_r)_{\lambda}=(\bigcap_{r\in\Gamma} T(H_r))_{\lambda}$. By theorem 2, we have $T(\bigcap_{r\in\Gamma} H_r)=\bigcap_{r\in\Gamma} T(H_r)$.

(4) $\forall\ \lambda \in [0,1], T(H^c)_{\lambda}=\bigcup_{\alpha<\lambda} H^c(\alpha)=\bigcup_{\alpha<\lambda}(H(1-\alpha))^c=(\bigcap_{\alpha<\lambda} H(1-\alpha))^c=(\bigcap_{1-\alpha>1-\lambda} H(1-\alpha))^c=(\bigcap_{\alpha>1-\lambda} H(\alpha))^c=(T(H)^{1-\lambda})^c=(T(H)^c)_{\lambda}$. By theorem 2, we have $T(H^c)=T(H)^c$.

References

1. Dubios, D. and Prade. H., *Fuzzy sets and systems*, Academic Press, New York, 1980.
2. Luo Chengzhong, *Introduction to fuzzy sets*, Beijing Normal University Press, 1989, (in Chinese).
3. Li Hongxing, Wang Peizhuang, *Fuzzy Mathematics*, Beijing National Defence Industry Press, 1994, (in Chinese).
4. Yuan Xuehai, E. S. Lee, *J. Math. Anal. Appl.* Vol. 177, No. 2(1993)p600—607.