Max-separable Functions and Fuzzy Sets[1]

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Abstract

Extremally separable functions in this note are functions of n variables, which are described as maximum or minimum of n functions of one variable. These functions are suggested as a tool for modelling some decision making problems in fuzzy environment. The problems are reformulated as optimization problems with an extermally separable objective function and inequality constraints with extermally separable functions in the left-hand sides. The conditions under which these problems are effectively solvable are discussed.

1. Introduction

We point out a special class of continuous functions of n variables, so called extremally separable functions as a tool for modelling certain decision making situations in fuzzy environment. Using these functions we formulate optimization problems, in which extremally separable functions occur both in the objective function and in the constraints. We consider a standard form of these problems (so called max-separable problems), show a motivational example of fuzzy decision making, which leads to solving of such an optimization problem, and using the results from [3] give conditions under which the problem is efectivelly solvable.

2. Max-separable functions and optimization problems

A function $\varphi : R^n \to R^1$ of the form $\varphi(x_1, \dots x_n) = \max_{j\in N} \varphi_j(x_j)$ or $\varphi(x_1, \dots x_n) = \min_{j\in N} \varphi_j(x_j)$, where $N = \{1, \dots, n\}$ and $\varphi_j : R^1 \to R^1$ will be called extremally separable (max-separable in the former and min-separable in the latter case).

We shall assume in the sequel that φ_j are continuous.

[1]This paper was presented at Fuzzy Workshop, which took place in Koovce (Slovakia), February 12-17,1995

An extremally separable optimization problem is an optimization problem with an extremally separable objective function and with a finite number of equality or inequality constraints with extremally separable functions. The following optimization problem is an example of an extremally separable optimization problem (it will be called max-separable optimization problem):

$$f(x_1, \dots x_n) \equiv \max_{j \in N} f_j(x_j) \to \min$$

subject to

$$r_i(x_1, \dots x_n) \equiv \max_{j \in N} r_{ij}(x_j) \geq b_i, \quad i \in S \equiv \{1, \dots, m\} \tag{P}$$

$$h_j \leq x_j \leq H_j, \quad j \in N$$

Such problems were studied e.g. in [3]. In this note we shall point out fuzzy optimization problems, which can be reduced to problems of the form (P). Further we shall summarize conditions under which problems of the form (P) can be effectively solved with the aid of a procedure suggested in [3].

Remark 2.1

It can be easily shown that other optimization problems with extremally separable functions in the left-hand sides of the constraints can be transformed to the standard form (P).

3. Fuzzy optimization problems reducible to (P)

In this paragraph we formulate some optimization problems in a fuzzy environment, which can be reduced to solving an optimization problem of the form (P). Let X_{ij}, $i \in N$ be cylindric fuzzy sets on R^n with membership functions $u_{X_{ij}}(x_1, \dots, x_n) \equiv r_{ij}(x_j)$ for all i, j. Let G_j, $j \in N$ be given cylindric fuzzy sets (fuzzy goals) on R^n with membership functions $u_{G_j}(x_1, \dots, x_n) \equiv g_j(x_j)$ for all $j \in N$. We want to find $(x_1, \dots, x_n)$ with a maximal membership value in $\overset{(F)}{\bigcap_{j \in N}} G_j$ ($\overset{(F)}{\bigcap}$ means the intersection of fuzzy sets), which satisfies at the same time the conditions that $(x_1, \dots, x_n)$ belongs to $\overset{(F)}{\bigcup_{j \in N}} X_{ij}$ ($\overset{(F)}{\bigcup}$ is the union of fuzzy sets) for all $i \in S$ at least with a prescribed membership value b_i and x_j lies within some prescribed bounds h_j, H_j for all $j \in N$. Transforming these formulations using membership functions and the standard theory of fuzzy sets (see e.g. [1], [2]), we obtain the following extremally separable optimization problem.

$$\max_{j \in N} f_j(x_j) \to \max$$

subject to

$$\max_{j \in N} r_{ij}(x_j) \geq b_i, \quad i \in S \tag{P1}$$

$$h_j \leq x_j \leq H_j, \quad j \in N,$$

where $f_j(x_j) \equiv 1 - g_j(x_j)$, i.e. $f_j(x_j)$ is the membership function of the complementary fuzzy set $\overline{G}_j$ and we want to find $(x_1, \ldots, x_n)$, which minimizes the membership to $\bigcup\limits_{j \in N}^{(F)} \overline{G}_j$.

We shall bring one motivating example showing us in which context such problems can arise. Let us assume that we have m customers (or clusters of customers) an n types of (substitutable) goods, the prices of which $p_1, \ldots, p_n$ should be determined under the condition that each of the customers must be satisfied with at least one of the prices on a level, which is greater or equal to a given number. In the other words we want that $(p_1, \ldots, p_n)$ belongs to the union of fuzzy sets X_{ij}, $j \in N$ (X_{ij} is the set of "satisfactory" prices p_j for the i-th customer). We want at the same time that p_j is as close as possible to some given "economic reasonable" price $\hat{p}_j$ and $p_j \in [h_j, H_j]$. If $r_{ij}(p_j)$ are membership functions of X_{ij} and we set e.g.

$$f_j(p_j) = \frac{|p_j - \hat{p}_j|}{H_j - h_j},$$

we obtain a problem of the form (P).

4. Assumptions for effective solubility of (P)

It arises a question under which assumptions concerning the properties of functions f_j, r_{ij}, we are able to suggest an effective algorithm for solving problems of the form (P). This question will be answered in this paragraph.

Let $V_{ij} \equiv \{x_j \in [h_j, H_j] | r_{ij}(x_j) \geq b_i\}$ for all $i \in S$, $j \in N$.

Assumption 1

There exists for each $j \in N$ a permutation $i_1^{(j)}, \ldots, i_m^{(j)}$ of indices $1, \ldots, m$ such that

$$V_{i_1^{(j)} j} \subseteq V_{i_2^{(j)} j} \subseteq \ldots \subseteq V_{i_m^{(j)} j}$$

Remark 4.1

(a) It can be easily shown that Assumption 1 is fulfilled if for each fixed j all functions r_{ij}, $i \in S$, are nonincreasing or all functions r_{ij}, $i \in S$ are nondecreasing.

(b) Assumption 1 is fulfilled if for each fixed $j \in N$ the graphs of functions $r_{ij}(x_j) - b_i$, $i \in S$ do not intersect each other.

(c) Let us remark that in the motivating example, it is reasonable to assume that r_{ij}'s are nonincreasing functions of p_j for all i, j.

The following theorems follow from the results in [3].

Theorem 4.1
Let **Assumption 1** be fulfilled and M be the set of feasible solutions of (P). Then $M = \emptyset \Longleftrightarrow \exists i_0 \in S$ such that $V_{i_0 j} = \emptyset$ for all $j = 1, \dots n$.

Theorem 4.2
Let **Assumption 1** be fulfilled and $M \neq \emptyset$. Let us set $f_j^{(i)} \equiv \min\{f_j(x_j) | x_j \in V_{ij}\}$ for all i, j such that $V_{ij} \neq \emptyset$. Let us set further: $f_{j(i)}^{(i)} \equiv \min_{j \in N} f_{j(i)}^{(i)}$, $L_j \equiv \{i \in S | j(i) = j\}$,

$$\overline{V}_j \equiv \begin{cases} [h_j, H_j] & \text{if } L_j = \emptyset \\ \bigcap_{i \in L_j} V_{ij} & \text{if } L_j \neq \emptyset \end{cases}$$

Let $x_j^{\text{opt}} \equiv \arg\min\{f_j(x_j) | x_j \in \overline{V}_j\}$ for all $j \in N$. Then $(x_1^{\text{opt}}, \dots, x_n^{\text{opt}})$ is an optimal solution of (P).

References

[1] Novák, V.: **Fuzzy Sets and Their Applications**, Adam Hilger, Bristol 1989.

[2] Zadeh, L.A.: Fuzzy Sets, Inf. & Control, **8**(1965), pp.338-353

[3] Zimmermann, K.: On max-separable Optimization Problems, Annals of Disc. Math. **19**(1984), pp.357-362