SOME PROPERTIES OF FUZZY AUTOMATA

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ABSTRACT. This paper describes some properties of fuzzy automata, especially, two variants of (Moore) type of fuzzy automaton. A representation theorem for output function of fuzzy automaton is derived. In addition, we prove that both variants of fuzzy automaton behave as a (Moore) type of fuzzy automaton.

1.THE (MOORE) TYPE OF FUZZY AUTOMATON

We consider the following (Moore) type of fuzzy automaton.

Definition 1.1. *A fuzzy automaton is a system*

$$\mathcal{A} = (S, \Lambda, p, \{F(\lambda : \lambda \in \Lambda\}, G)$$

where

$S = \{s_1, s_2, \dots, s_n\}$ *is a finite set of states,*

Λ *is a finite set of inputs,*

$p \underset{\sim}{\subset} S$ *is a fuzzy set called a fuzzy initial state,*

G *is a set of final states, i.e.* $G \subseteq S$,

$F(\lambda) \underset{\sim}{\subset} S \times S$ *is a fuzzy transition matrix of order n, i.e. a fuzzy relation in S.*

The elements of $F(\lambda)=\|F_{s,t}(\lambda)\|$ are the values of a fuzzy transition function; i.e. a membership function of a fuzzy set in $S \times \Lambda \times S$

$$F\colon S \times \Lambda \times S \to [0, 1].$$

That is to say, for s,t $\in S$ and $\lambda \in \Lambda$, $F(s, \lambda, t)$ is the grade of transition from state s to state t when the input is λ.

Every fuzzy set A of S is called a fuzzy state of $\mathcal{A}$. If an input signal $\lambda \in \Lambda$ is accepted by $\mathcal{A}$, the present fuzzy state A of $\mathcal{A}$ will be changed to the state $B = A \circ F(\lambda)$, where "$\circ$" is a composition rule of fuzzy relations (e.g. a minimax product).

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The fuzzy transition function F can be extended to fuzzy transition function

$$F^*\colon S \times \Lambda^* \times S \to [0,1]$$

so that for every $\lambda = \lambda_1\lambda_2 \dots \lambda_n \in \Lambda^*$ the following diagram commutes

$$\begin{array}{ccc} S \times \Lambda \times S & \xrightarrow{F} & [0,1] \\ {\scriptstyle u}\big\downarrow & & \big\| \\ S \times \Lambda^* \times S & \xrightarrow{F^*} & [0,1]. \end{array}$$

For $\lambda = \lambda_1\lambda_2 \dots \lambda_n \in \Lambda^*$ the fuzzy transition matrix is defined as a composition of fuzzy relations, $F^*(\lambda) = F(\lambda_1) \circ F(\lambda_2) \circ \dots F(\lambda_n)$.

A principal identification of fuzzy automaton is provided by its output function

$$f_A\colon \Lambda^* \to [0,1]$$

so that for $\lambda = \lambda_1\lambda_2 \dots \lambda_n \in \Lambda^*$,

$$f_A(\lambda) = p \circ F(\lambda_1) \circ \dots \circ F(\lambda_n) \circ G = \bigvee_{s\in S}(\bigvee_{z\in S}(p(z) \wedge F(\lambda)(z,s)) \wedge G(s)).$$

Clearly, $f_A(\lambda)$ is designated as the grade of transition of $\mathcal{A}$, when $\mathcal{A}$ starts with the initial state p to enter into a state in G after scanning the input sequence λ. Then an input sequence $\lambda \in \Lambda^*$ is said to be accepted by $\mathcal{A}$ with a grade $f_A(\lambda)$.

Here we recall that a (classical) Moore type automaton is a system

$$\mathcal{B} = (S, \Lambda, p, d, G),$$

where S, Λ and G have the same meaning as in Definition 1.1., $p \in S$ is the initial state, $d\colon S \times \Lambda \to S$ is a transition function.

Let $\mathcal{B} = (S, \Lambda, p, d, G)$ is a Moore type automaton; think of $\mathcal{B}$ as being "in" exactly one of its states at any moment and being allowed to change the state s it is currently in by reading an input signal $\lambda \in \Lambda$ and then moving to the state d(s,λ).

The transition function $d\colon S \times \Lambda \to S$ can be extended to

$$d^*\colon S \times \Lambda^* \to S$$

so that for $\lambda = \lambda_1\lambda_2 \dots \lambda_n \in \Lambda^*$ the following diagram commutes

$$\begin{array}{ccc} S \times \Lambda & \xrightarrow{d} & S \\ {\scriptstyle u}\big\downarrow & & \big\| \\ S \times \Lambda^* & \xrightarrow{d^*} & S. \end{array}$$

Now, given $\lambda = \lambda_1\lambda_2\ldots\lambda_n \in \Lambda^*$,the $\mathcal{B}$ can process λ as follows: $\mathcal{B}$ begins in the initial state p and then inputs the characters of λ one by one, each time changing the state; it is according to the value of the transition function for the current state and the character of λ being input.

Any Moore type automaton $\mathcal{B}$ provides an output function

$$f_B\colon \Lambda^* \to \{0,1\}$$

so that for $\lambda = \lambda_1\lambda_2\ldots\lambda_n \in \Lambda^*$,

$$\begin{array}{ccc} \Lambda^* & \xrightarrow{f_B} & \{0,1\} \\ {\scriptstyle u}\big\downarrow & & \big\uparrow{\scriptstyle \chi_G} \\ S \times \Lambda^* & \xrightarrow{d^*} & S \end{array}$$

$$f_B(\lambda) = (\chi_G \circ d^* \circ u)(\lambda) = \chi_G(d^*(p,\lambda)) = 1 \text{ iff}$$

$$d^*(p,\lambda) := d(d^*(p,\lambda_1\lambda_2\ldots\lambda_{n-1}),\lambda_n) \in G.$$

It should be observed that every Moore type automaton behaves as a fuzzy automaton. Indeed, the following proposition holds.

Proposition 1.2. *Let $\mathcal{B} = (S,\Lambda,p,d,G)$ be a Moore type automaton. Then there exists a fuzzy automaton $F(\mathcal{B}) = (S,\Lambda,p,\{F(\lambda)\colon \lambda \in \Lambda\},G)$ so that*

$$f_{F(B)}(\lambda) = f_B(\lambda) \text{ for all } \lambda \in \Lambda^*,$$

where $f_{F(B)}(\lambda) = \bigvee_{s\in S}(\bigvee_{z\in S}(p(z)\wedge F(\lambda)(z,s))\wedge G(s))$ *and* $f_B(\lambda) = \chi_G(d^*(p,\lambda))$.

Proof. Let $\lambda \in \Lambda$ and let us define a fuzzy matrix $F(\lambda) \underset{\sim}{\subset} S \times S$ such that

$$(1) \qquad F(\lambda)(s,t) = \begin{cases} 1, & \text{if d(s,}\lambda\text{)=t} \\ 0, & \text{if d(s,}\lambda\text{)}\neq\text{t.} \end{cases}$$

We observe at first that the rule (1) holds even for $\lambda \in \Lambda^*$ and not for elements of Λ only. This may be proved easily by using the induction principle on the length $\|\lambda\|$ of $\lambda \in \Lambda$.

Then it could be verified without any problem that $f_{F(B)}(\lambda) = f_B(\lambda)$ for all $\lambda \in \Lambda^*$. □

2. Variants of fuzzy automata

We shall now begin considering special variants of fuzzy automata. Using definition 1.1., we introduce two special variants of (Moore) type of fuzzy automaton and review some results along this line.

Definition 2.1. *A first variant of fuzzy automaton is a system*

$$\mathcal{C} = (S, \Lambda, p, d, G \underset{\sim}{\subset} S)$$

where

$S = \{s_1, s_2, \dots, s_n\}$ *is a finite set of states,*
Λ *is a finite set of inputs,*
$p \in S$ *is the initial state,*
$d\colon S \times \Lambda \to S$ *is a transition function,*
$G \underset{\sim}{\subset} S$ *is a fuzzy set called a fuzzy final state.*

This variant of fuzzy automaton provides an output function

$$f_C\colon \Lambda^* \to [0,1]$$

so that for $\lambda = \lambda_1\lambda_2\dots\lambda_n \in \Lambda^*$,

$$\begin{array}{ccc} \Lambda^* & \xrightarrow{f_C} & [0,1] \\ {\scriptstyle u}\big\downarrow & & \big\uparrow{\scriptstyle G} \\ S \times \Lambda^* & \xrightarrow{d^*} & S \end{array}$$

$$f_C(\lambda) = (G \circ d^* \circ u)(\lambda) = G(d^*(p, \lambda)).$$

The first variant of fuzzy automaton behaves as a (Moore) type of fuzzy automaton. The following proposition illustrates it.

Proposition 2.2. *Let $\mathcal{C} = (S, \Lambda, p, d, G \underset{\sim}{\subset} S)$ be a first variant of (Moore) type of fuzzy automaton. Then there exists a fuzzy automaton*
$F(\mathcal{C}) = (S, \Lambda, p, \{F(\lambda)\colon \lambda \in \Lambda\}, G)$ so that

$$f_{F(C)}(\lambda) = f_C(\lambda) \text{ for all } \lambda \in \Lambda^*,$$

where $f_{F(C)}(\lambda) = \bigvee\limits_{s \in S} \big(\bigvee\limits_{z \in S} (p(z) \wedge F(\lambda)(z,s)) \wedge G(s)\big)$ and $f_C(\lambda) = G(d^*(p,\lambda))$.

Proof. According to 1.2. we define a fuzzy matrix $F(\lambda) \underset{\sim}{\subset} S \times S$. Then it may be proved that $f_{F(C)}(\lambda) = f_C(\lambda)$ for all $\lambda \in \Lambda^*$. □

Definition 2.3. *A second variant of fuzzy automaton is a system*

$$\mathcal{D} = (S, \Lambda, p \underset{\sim}{\subset} S, d, G)$$

where

$S = \{s_1, s_2, \dots, s_n\}$ *is a finite set of states,*
Λ *is a finite set of inputs,*
$p \underset{\sim}{\subseteq} S$ *is a fuzzy set called a fuzzy initial state,*
$d\colon \Lambda \times S \to S$ *is a transition function,*
$G \subseteq S$ *is a set of final states.*

Analogously, this variant of fuzzy automaton provides an output function

$$f_D\colon \Lambda^* \to [0,1]$$

so that for $\lambda = \lambda_1\lambda_2\dots\lambda_n \in \Lambda^*$,

$$\begin{array}{ccc} \Lambda^* & \xrightarrow{f_D} & [0,1] \\ {\scriptstyle u}\big\downarrow & & \big\uparrow{\scriptstyle h} \\ \prod\limits_{s\in S}(S\times[0,1])\times\Lambda^* & \xrightarrow{d_1^*} & \prod\limits_{s\in S}(S\times[0,1]) \end{array}$$

$$f_D(\lambda) = (h \circ d_1^* \circ u)(\lambda) = (h \circ d_1^*)((s, p(s)_{s\in S}), \lambda) = h(d^*(s,\lambda), p(s)_{s\in S}) = \\ = \bigvee_{s\in S} (G(d^*(s,\lambda)) \wedge p(s)).$$

The following proposition describes that the second variant of fuzzy automaton behaves as a (Moore) type of fuzzy automaton.

Proposition 2.4. *Let $\mathcal{D} = (S, \Lambda, p \underset{\sim}{\subseteq} S, d, G)$ be a second variant of (Moore) type of fuzzy automaton. Then there exists a fuzzy automaton*
$F(\mathcal{D}) = (S, \Lambda, p, \{F(\lambda)\colon \lambda \in \Lambda\}, G)$ so that

$$f_{F(D)}(\lambda) = f_D(\lambda) \text{ for all } \lambda \in \Lambda^*,$$

where $f_{F(D)}(\lambda) = \bigvee_{s\in S} (\bigvee_{z\in S} (p(z) \wedge F(\lambda)(z,s)) \wedge G(s))$ and
$f_D(\lambda) = \bigvee_{s\in S} (G(d^*(s,\lambda)) \wedge p(s))$.

Proof. Analogously as in proof of 1.2., we define a fuzzy matrix $F(\lambda) \underset{\sim}{\subseteq} S \times S$. Then it may be proved that $f_{F(D)}(\lambda) = f_D(\lambda)$ for all $\lambda \in \Lambda^*$. □

References

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