ADDITION OF L–R FUZZY NUMBERS[1]

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ABSTRACT. Triangular norm-based addition of L–R fuzzy numbers is investigated. Recent results with analytical form of output sum of L–R fuzzy numbers are recalled. A necessary and sufficient condition ensuring the output sum in the form presented by Hong and Hwang is introduced in the case of strict t-norms. For nilpotent t-norms, a sufficient condition including all known sufficient conditions is introduced, too. Some possible applications are indicated.

1. Introduction

Fuzzy numbers and their addition belong to the theoretical basis of several intelligent technologies based on the vague input data. The above fact has increased the interest on theoretical aspects of the topic [1,2,3,4,6,9,11,12,13,15,16]. In this paper we deal with t-norm based addition (T-sum) of L–R fuzzy numbers, especially we look for the conditions ensuring the exact computation of the output sum.

Definition 1. A fuzzy number A is a fuzzy subset of the real line $\mathbb{R}$ with a continuous, compactly supported, unimodal and normalized membership function. Following Dubois and Prade [1], a fuzzy number A is a so called L–R fuzzy number, $A = (a, \alpha, \beta)_{LR}$, if the corresponding membership function satisfies for all $x \in \mathbb{R}$

$$A(x) = \begin{cases} L\left(\frac{a-x}{\alpha}\right), & \text{for } a - \alpha \leqq x \leqq a, \\ R\left(\frac{x-a}{\beta}\right), & \text{for } a \leqq x \leqq a + \beta, \\ 0, & \text{else.} \end{cases}$$

where $a \in \mathbb{R}$ is the peak of A, $\alpha > 0$ and $\beta > 0$ is the left and the right spread, respectively, and L and R are decreasing continuous functions from $[0, 1]$ to $[0, 1]$

[1] This paper was presented at Fuzzy Workshop, which took place in Kočovce (Slovakia), February 12-17,1995

such that $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. Recall that L and R is called the left and the right shape function, respectively.

Let T be a given t-norm and let A_1 and A_2 be two fuzzy numbers. Then their T-sum is defined by

$$A_1 \oplus_T A_2(z) = \sup_{x+y=z} \left(T\left(A_1(x), A_2(y)\right)\right), \qquad z \in \mathbb{R}.$$

Recall that since Dubois and Prade [1] it is known that the left (the right) side of the output sum $A \oplus_T B$ depends only on T and the left (the right) sides of incomming L–R fuzzy numbers A, B. Put $A_i = (a_i, \alpha_i, \beta_i)_{LR}$, $i = 1, 2$. Then $A_1 \oplus_{T_M} A_2 = (a_1+a_2, \alpha_1+\alpha_2, \beta_1+\beta_2)_{LR}$ and $A_1 \oplus_{T_W} A_2 = (a_1+a_2, \max(\alpha_1, \alpha_2), \max(\beta_1, \beta_2))_{LR}$ for arbitrary shapes L,R, where T_M and T_W is the strongest and the weakest t-norm, respectively. Further, if $T_1 \leqq T_2$ then $A \oplus_{T_1} B \leqq A \oplus_{T_2} B$ for arbitrary fuzzy numbers A, B. For more details see [15].

2. T-sums of L–R fuzzy numbers

The exact output of a T-sum of some L–R fuzzy numbers in an analytical form was found only for some special cases. For continuous Archimedean t-norm T with additive generator f, the most general published result is due to Hong and Hwang [9], generalizing the earlier results of Fullér and Keresztfalvi [4].

Theorem 1. *[9] Let T be an Archimedean t-norm with additive generator f and let $A_i = (a_i, \alpha, \beta)_{LR}$, $i = 1, \dots, n, \dots$, be L–R fuzzy numbers. If L and R are concave functions and f is a convex function, then the membership function of T-sum $B_n = A_1 \oplus_T \cdots \oplus_T A_n$ is given by*

$$B_n(z) = \begin{cases} f^{[-1]}\left(nf\left(L\left(\frac{S_n - z}{n\alpha}\right)\right)\right), & \text{for } S_n - n\alpha \leqq z \leqq S_n, \\ f^{[-1]}\left(nf\left(R\left(\frac{z - S_n}{n\beta}\right)\right)\right), & \text{for } S_n \leqq z \leqq S_n + n\beta, \\ 0, & \text{else.} \end{cases}$$

where $f^{[-1]}$ is the pseudo-inverse of f, defined by

$$f^{[-1]}(y) := \begin{cases} f^{-1}(y), & \text{if } y \in [0, f(0)], \\ 0, & \text{if } y \in [f(0), \infty]. \end{cases}$$

and $S_n = a_1 + \cdots + a_n$.
Let $S := \sum_{i=1}^{\infty} a_i$ exists and be finite. Then

$$\lim_{n\to\infty} B_n(z) = \begin{cases} f^{[-1]}\left(\frac{S-z}{\alpha} f'_-(1) L'_+(0)\right), & \text{for } z \leqq S, \\ f^{[-1]}\left(\frac{z-S}{\beta} f'_-(1) R'_+(0)\right), & \text{for } z \geqq S. \end{cases}$$

For special case of the product t-norm T_P, Triesch [16] obtained a more general result. Namely, the concavity of both $\log L$ and $\log R$ ensure the conclusions of Theorem 1. Recently, Hong [6] showed that Triesch conditions are not only sufficient, but also necessary.

In a very recent paper of Mesiar [13], even more general conditions as those in [9] are considered.

Theorem 2. *[13] The conclusions of Theorem 1 remain true if both $f \circ L$ and $f \circ R$ are convex functions.*

Recall that product t-norm T_P has the additive generator $f(x) = -\log x$ and thus Triesch's result [16] is a corollary of Mesiar's Theorem 2.
For strict t-norm T, we have shown also the necessity of Mesiar's conditions.

Theorem 3. *For a strict t-norm T the conclusions of Theorem 1 remain valid if and only if both $f \circ L$ and $f \circ R$ are convex.*

Now, the result of Hong [6] is a corollary of Theorem 3. However, the case of nilpotent t-norms is more complicated.

Example 1. Let T be a t-norm generated by the additive generator f,

$$f(x) = \begin{cases} 1 - 8x^2, & \text{if } x \in [0, \frac{1}{4}], \\ \frac{2}{3}(1-x), & \text{if } x \in [\frac{1}{4}, 1]. \end{cases}$$

Note that f is not a convex function (neither concave). However, for arbitrary linear fuzzy numbers ($L(x) = R(x) = 1 - x$) $A_i = (a_i, \alpha, \beta)$ the conclusions of Theorem are still valid! Put, e.g., $A_1 = A_2 = (0, 1, 1)$, $B = A_1 \oplus_T A_2$. Then for $z \geqq 0$ it is

$$B(z) = \begin{cases} 1 - z, & \text{if } 0 \leqq z \leqq \frac{3}{4}, \\ \left(\frac{3-2z}{24}\right)^{\frac{1}{2}}, & \text{if } \frac{3}{4} \leqq z \leqq \frac{3}{2}, \\ 0 & \text{if } z \geqq \frac{3}{2}. \end{cases}$$

Note also that if $A_i = (0, 1, 1)$, $i = 1, \ldots, n$, $n \geqq 2$, then $A_1 \oplus_T \cdots \oplus_T A_n = B$ and $B \oplus_T B = B$.

The above example shows that for nilpotent t-norms even the Mesiar's conditions can be strengthened. We have the following sufficient condition.

Theorem 4. *Let T be a nilpotent t-norm with the normed additive generator f. Then the conclusions of Theorem 1 are true if both $K_1 = f \circ L$ and $K_2 = f \circ R$ fulfill the following two conditions (for $i = 1, 2$):*

(1) *K_i is convex on $[0, K_i^{-1}\left(\frac{1}{2}\right)]$*
(2) *on $[K_i^{-1}\left(\frac{1}{2}\right), 1]$ K_i is not below the left tangent of K_i in point $\left(K_i^{-1}\left(\frac{1}{2}\right), \frac{1}{2}\right)$, i.e., $K_i(x) \geqq \frac{1}{2} + K_i'^{-}\left(K_i^{-1}\left(\frac{1}{2}\right)\right)\left(x - K_i^{-1}\left(\frac{1}{2}\right)\right)$, $x \in [K_i^{-1}\left(\frac{1}{2}\right), 1]$.*

Recall that if we want to add at most n (n is fixed) L–R fuzzy numbers then the condition (2) can be relaxed to

$$K_i(x) \geqq \frac{n}{2} - (n-1)K_i\left(\frac{nK_i^{-1}\left(\frac{1}{2}\right) - x}{n-1}\right) \quad \text{for } x \in [K_i^{-1}\left(\tfrac{1}{2}\right), 1], i = 1, 2.$$

Note that the limit (the supremum) of the right sides of the above inequality leads just to the condition (2) of the Theorem 4. Further note that the Example 1 fulfils the requirements of the Theorem 4.

3. Applications

i) Fullér's theorem [2] on the law of large numbers is valid for each t-norm T with additive generator$f$ and L–R fuzzy numbers such that both $f \circ L$ and $f \circ R$ are convex. Consequently the same is true, for given shapes L,R, for each t-norm $T^* \leqq T$. Note that Fullér's result concerns Hamacher's t-norm $T_0^H(x,y) = \frac{xy}{x+y-xy}$ and linear fuzzy numbers, i.e., $f(x) = \frac{1-x}{x}$, $L(x) = R(x) = 1-x$, and hence $f \circ L(x) = f \circ R(x) = \frac{x}{1-x}$ is convex.

ii) The compositional rule of inference under triangular norms was stated by Fullér and Zimmermann [5] for a t-norm T with twice differentiable concave functions ϕ_1, ϕ_2. This result was generalized by Hong and Hwang [8] for convex f and concave ϕ_1, ϕ_2. Using our results, we guarante the validity of Fullér - Zimmermann composition law for $f \circ \phi_1$ and $f \circ \phi_2$ convex (this is necessary condition for strict t-norm T, too). If T is nilpotent, then we can even relax the convexity of $f \circ \phi_1$ and $f \circ \phi_2$ to their convexity on $[0, (f \circ \phi_i)^{-1}(\frac{1}{2})]$ together with $f \circ \phi_i(x) + f \circ \phi_i\left(2\,(f \circ \phi_i)^{-1}(\frac{1}{2}) - x\right) \geqq 1$, $x \geqq (f \circ \phi_i)^{-1}(\frac{1}{2}) \geqq \frac{1}{2}$, $i = 1, 2$.

iii) The T-sum of fuzzy numbers of cummulative type with common spread and shape [10] can be treated in a way similar to the T-sum of L–R fuzzy numbers.

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