Extensions of Boolean functions to T-tribes of fuzzy sets [1]

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Abstract

We introduce T-admissible functions on T-tribes as "all possible" fuzzy extensions of (n-ary) Boolean functions. For the Frank family of T-norms, we give a characterization of T-admissible functions and a characterization of T-tribes. The paper summarizes recent results of [1, 4, 6, 7, 8] in a unified context.

1 Basic notions

The concept of T-tribes on a universe, i. e., a nonempty crisp set X, where T is a t-norm and the elements of the T-tribes are fuzzy subsets of X, was introduced in [1, 3] in order to have a proper generalization of the classical σ-algebras. Here we investigate the question which functions can be composed from the t-norm T and a fuzzy complement, i. e., with respect to which functions every T-tribe is closed.

Let us first recall some basic notions and facts from [1, 9]. A *t-norm* (triangular norm) is a function $T : [0,1]^2 \to [0,1]$ which is commutative, associative, monotone in each component, and satisfies the boundary condition $T(1,x) = x$. Throughout this paper, T denotes a t-norm.

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We shall deal mostly with Frank t-norms F_s. For $s \in (0,\infty)\setminus\{1\}$, they are defined by the formula

$$F_s : (x,y) \mapsto \log_s\left(1 + \frac{(s^x-1)(s^y-1)}{s-1}\right).$$

The limit cases are:

$$F_0 = T_{\mathbf{M}} : (x,y) \mapsto \min(x,y) \text{ (minimum } t\text{-norm)},$$

$$F_\infty = T_{\mathbf{L}} : (x,y) \mapsto \max(x+y-1,0) \text{ (Łukasiewicz } t\text{-norm)},$$

$$F_1 = T_{\mathbf{P}} : (x,y) \mapsto x \cdot y \text{ (product } t\text{-norm)}.$$

For fuzzy subsets of X, say $f, g \in [0,1]^X$, we extend a given t-norm T pointwise, i. e.,

$$T(f,g)(x) = T(f(x), g(x)).$$

This operation may be considered as an "intersection" of fuzzy sets. We define the complement by $x \mapsto 1-x$. Restricted to crisp sets, i. e., to characteristic functions, we obtain the usual set-theoretical operations.

The t-norm T can be naturally extended to a function of finitely or countably many variables, denoted $T_{m\in M}$.

Definition 1.1 : A *T-tribe* on X is a collection $\mathscr{T} \subseteq [0,1]^X$ such that:

1. $1 \in \mathscr{T}$,
2. if $f \in \mathscr{T}$ then $1-f \in \mathscr{T}$,
3. if $(f_n)_{n\in\mathbf{N}} \subseteq \mathscr{T}$ then $T_{n\in\mathbf{N}} f_n \in \mathscr{T}$.

We denote by $\mathbf{1}_B$ the characteristic function of a crisp set $B \subset X$. For each T-tribe $\mathscr{T}$ on X, we define

$$\mathscr{T}^\vee = \{Y \subseteq X \mid \mathbf{1}_Y \in \mathscr{T}\},$$

which is a σ-algebra on X, showing that T-tribes are proper generalizations of σ-algebras.

The following results are taken from [1].

Theorem 1.2 :

1. *For each $s \in (0,\infty)$, every F_s-tribe is a $T_{\mathbf{L}}$-tribe* ([1, Proposition 2.7]).
2. *Every $T_{\mathbf{L}}$-tribe is a $T_{\mathbf{M}}$-tribe* ([1, Proposition 2.7]).
3. *All elements of a $T_{\mathbf{L}}$-tribe $\mathscr{T}$ are $\mathscr{T}^\vee$-measurable* ([3],[1, Proposition 3.2]).

Since we may view the elements of $[0,1]$ also as constant functions on a singleton domain, we define a *T-tribe of constants* as a set $K \subseteq [0,1]$ such that

1. $1 \in K$,
2. if $r \in K$ then $1 - r \in K$,
3. if $(r_n)_{n\in\mathbf{N}} \subseteq K$ then $T_{n\in\mathbf{N}}\, r_n \in K$.

In particular, $T_{\mathbf{M}}$-tribes of constants are all closed subsets of $[0,1]$ which are symmetric with respect to $1/2$ and contain 0 and 1. The only $T_{\mathbf{L}}$-tribes of constants are

$$K_n = \left\{\frac{i}{n} \mid i = 0, \dots, n\right\}$$

for $n \in \mathbf{N}$, and

$$K_\infty = [0,1].$$

For $s \in (0,\infty)$, there are only two F_s-tribes of constants, $K_1 = \{0,1\}$ and K_∞.

A set $G \subseteq [0,1]^X$ *T-generates* a T-tribe $\mathscr{T}$ if $\mathscr{T}$ is the smallest T-tribe on X containing G.

Example 1.3 : Let $n \in \mathbf{N}$, $n \geq 2$, and $G = \{1/n\}$. Then

1. G $T_{\mathbf{M}}$-generates $\{0, 1, 1/n, (n-1)/n\}$,
2. G $T_{\mathbf{L}}$-generates K_n,
3. for $s \in (0,\infty)$, G F_s-generates K_∞.

2 Characterizations of T-admissible functions

For $n \in \mathbf{N}$, there are 2^{2^n} n-ary Boolean functions defined on $\{0,1\}^n$. We ask here which of their extensions to $[0,1]^n$ are applicable to elements of T-tribes (so that the result belongs to the same T-tribe). We find that these are so called T-admissible functions, introduced in [4, 8].

Definition 2.1 : An n-ary *T-admissible function* is a function $a : [0,1]^n \to [0,1]$ which belongs to the T-tribe $\mathscr{T}$ on $[0,1]^n$ T-generated by $\{\mathrm{pr}_1, \dots, \mathrm{pr}_n\}$, where $\mathrm{pr}_i : (x_1, \dots, x_n) \mapsto x_i$ is the projection onto the i-th component, $i = 1, \dots, n$.

If T is (Borel) measurable, all T-admissible functions are measurable. The following is a full characterization of T-admissible functions:

Proposition 2.2 (composition principle, cf. [8]): *A function $a : [0,1]^n \to [0,1]$ is T-admissible if and only if for each T-tribe $\mathscr{T}$ and each $f_1, \dots, f_n \in \mathscr{T}$ the composition $a(f_1, \dots, f_n) : x \mapsto a(f_1(x), \dots, f_n(x))$ is an element of $\mathscr{T}$.*

Therefore T-admissible functions are exactly the functions such that each T-tribe is closed with respect to them. Note that a necessary condition for the T-admissibility of a is

$$\text{(A)} \qquad a(x_1, \ldots, x_n) \in \{0,1\} \text{ whenever } \{x_1, \ldots, x_n\} \subseteq \{0,1\}.$$

This means that a restricted to Boolean arguments $\{0,1\}$ is a Boolean function. Thus T-admissible functions may be considered as "all possible" fuzzy generalizations of Boolean functions. As a consequence, the only nullary T-admissible functions are 0 and 1.

Corollary 2.3 : *The T-tribe $\mathscr{A}$ of n-ary T-admissible functions is closed with respect to composition, i. e., for all $a, b_1, \ldots, b_n \in \mathscr{A}$ we have*

$$a(b_1, \ldots, b_n) \in \mathscr{A},$$

where $a(b_1, \ldots, b_n) : \mathbf{x} \mapsto a(b_1(\mathbf{x}), \ldots, b_n(\mathbf{x})) \quad (\mathbf{x} \in [0,1]^n)$.

Theorem 2.4 : *(i) The unary $T_{\mathbf{M}}$-admissible functions are $\{0,\ 1,\ pr_1,\ pr'_1,\ pr_1 \wedge pr'_1,\ pr_1 \vee pr'_1\}$, where $pr'_1 = 1 - pr_1$.*

Let $n \in \mathbf{N}$. All n-ary $T_{\mathbf{M}}$-admissible functions are piecewise linear and they are uniquely determined by their values on $M = \{0, 1/2, 1\}^n$. The values on M belong to $\{0, 1/2, 1\}$. A consequence: There are finitely many n-ary $T_{\mathbf{M}}$-admissible functions.

(ii) A function $a : [0,1]^n \to [0,1]$ is $T_{\mathbf{L}}$-admissible if and only if it is measurable and the value $a(x_1, \ldots, x_n)$ belongs to the $T_{\mathbf{L}}$-tribe of constants $T_{\mathbf{L}}$-generated by $\{x_1, \ldots, x_n\}$ whenever $\{x_1, \ldots, x_n\} \subseteq \mathbf{Q}$. In particular, a unary function $a : [0,1] \to [0,1]$ is $T_{\mathbf{L}}$-admissible if and only if it is measurable and

$$\text{(K)} \quad a(i/n) \in K_n \textit{ for all } n \in \mathbf{N} \textit{ and } i \in \{0, \ldots, n\} \textit{ with } \gcd(i,n) = 1,$$

(gcd denotes the greatest common divisor).

(iii) Let $s \in (0, \infty)$. A function $a : [0,1]^n \to [0,1]$ is F_s-admissible if and only if it is measurable and satisfies (A). *In particular, all measurable t-norms are F_s-admissible. A consequence: An F_s-tribe is a T-tribe for any measurable t-norm T.*

Corollary 2.5 : *Every element of a $T_{\mathbf{L}}$-tribe $\mathscr{T}$ is a uniform limit of a monotone sequence of elements of $\mathscr{T}$ with countable range.*

For $s \in (0, \infty)$, every element of an F_s-tribe $\mathscr{T}$ is a uniform limit of a monotone sequence of elements of $\mathscr{T}$ with finite range.

3 Characterizations of T-tribes

The characterization of T-admissible functions gives us a tool for the characterization of T-tribes. Before treating this problem in its full generality, we introduce the class of semigenerated T-tribes. This class is smaller, but easier to work with, and sufficiently general for many applications.

Definition 3.1 : A collection $\mathscr{T} \subseteq [0,1]^X$ is called a *semigenerated T-tribe* if there exist a σ-algebra $\mathscr{B}$ of subsets of X, a sequence $(C_n)_{n\in\mathbf{N}}$ of T-tribes of constants and a countable $\mathscr{B}$-partition $(X_n)_{n\in\mathbf{N}}$ of X such that

$$\mathscr{T} = \{f \in \prod_{n\in\mathbf{N}} C_n^{X_n} \mid f \text{ is } \mathscr{B}\text{-measurable}\}.$$

Obviously, a semigenerated T-tribe is a T-tribe. Moreover, if T_1 and T_2 are t-norms and if a T_1-tribe is a semigenerated T_2-tribe, then it is also a semigenerated T_1-tribe.

In particular, a collection $\mathscr{T} \subseteq [0,1]^X$ is a semigenerated $T_{\mathbf{L}}$-tribe if there exist a σ-algebra $\mathscr{B}$ of subsets of X and a disjoint sequence $(X_n)_{n\in\mathbf{N}} \subseteq \mathscr{B}$ such that

$$\mathscr{T} = \{f \in [0,1]^X \mid f \text{ is } \mathscr{B}\text{-measurable, } f(x) \in K_n \text{ whenever } x \in X_n\}.$$

For $s \in (0,\infty)$, a collection $\mathscr{T} \subseteq [0,1]^X$ is a semigenerated F_s-tribe if and only if there exist a σ-algebra $\mathscr{B}$ of subsets of X and an $X_1 \in \mathscr{B}$ such that $\mathscr{T}$ is the set of all $\mathscr{B}$-measurable elements of $\{0,1\}^{X_1} \times [0,1]^{X\setminus X_1}$.

The following theorem asserts that any countable subset of an F_s-tribe, resp. $T_{\mathbf{L}}$-tribe, is contained in a semigenerated sub-F_s-tribe, resp. sub-$T_{\mathbf{L}}$-tribe.

Theorem 3.2 : *Let $s \in (0,\infty]$. Every F_s-tribe F_s-generated by a countable set is semigenerated.*

The full characterization of F_s-tribes, resp. $T_{\mathbf{L}}$-tribes, is given using the $T_{\mathbf{L}}$-tribes of constants K_n, $n \in \mathbf{N}$.

Theorem 3.3 : *(i) A collection $\mathscr{T} \subseteq [0,1]^X$ is a $T_{\mathbf{L}}$-tribe if and only if there exist a σ-algebra $\mathscr{B}$ of subsets of X and a sequence $(\nabla_n)_{n\in\mathbf{N}}$ of σ-filters in $\mathscr{B}$ with $\nabla_m \subseteq \nabla_n$ whenever n is a divisor of m, such that*

$$\mathscr{T} = \{f \in [0,1]^X \mid f \textit{ is } \mathscr{B}\textit{-measurable, } f^{-1}(K_n) \in \nabla_n \textit{ for all } n \in \mathbf{N}\}.$$

(ii) Let $s \in (0,\infty)$. A collection $\mathscr{T} \subseteq [0,1]^X$ is an F_s-tribe if and only if there exist a σ-algebra $\mathscr{B}$ of subsets of X and a σ-filter ∇_1 in $\mathscr{B}$ such that

$$\mathscr{T} = \{f \in [0,1]^X \mid f \textit{ is } \mathscr{B}\textit{-measurable, } f^{-1}(\{0,1\}) \in \nabla_1\}.$$

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