GENERATED FUZZY SUBGROUPS AND GENERATING SYSTEM OF CONJUGATE FUZZY SUBGROUPS

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Abstract

In this paper, a method of generating fuzzy subgroups by fuzzy subsets was given, using the theory of nest sets. Based on the method, we discussed the relationship between fuzzy generating sysyem of fuzzy subgroups H and fuzzy generating system of fuzzy conjugate subgroup H_g^* determined by H and a element g.

Keywords: fuzzy subgroup; nest set; generated fuzzy subgroup; conjugate fuzzy subgroup.

Since A. Rosenfeld introduced the concept of fuzzy set into group theory [5], mathematicians have studied, from different direction, fuzzy groups in a deep-going way. V. N. Dixit et al. have investegated fuzzy subgroup <A> generated by fuzzy subset A with condition CardImA $<\infty$, and obtained some reasonable results. But the case CardImA=$+\infty$ has not been discussed. In this paper, taking the stratum structures of fuzzy sets as the point of departure, and using the theory of nest set, we deeply studied fuzzy subgroup <A> generated by general fuzzy subset A, which is totally different from [1], and discussed the generating system of conjugate fuzzy subgroups.

<u>Definition 1</u> Let G be a group, and A a fuzzy subset of G. The minimal fuzzy subgroup in G containing A is called the generated fuzzy subgroup by A, and written as <A>, and A is called a fuzzy generating

system of <A>.

It is clear that A is a fuzzy subgruop iff $<A>=A$.

The following theorem gives the method of structuring generated fuzzy subgroups.

Theorem 1 Let A be a fuzzy subset of group G. Then

$$<A>=\bigcup_{0\leq\lambda\leq 1}\lambda<A_\lambda>,$$

where, A_λ is λ-cut set of A, $<A_\lambda>$ is crisp subgroup generated by A_λ in G.

Before Theorem 1 is proved, we introduce some preparatory knowledge.

Definition 2 [2] The mapping

$$H: [0,1]\rightarrow P(X): \lambda \mid\rightarrow H(\lambda)$$

is called a nest set on X, if

$$\lambda_1<\lambda_2 \Longrightarrow H(\lambda_1)\supset H(\lambda_2).$$

If H is a nest sets on X, then

$$T(H) = \bigcup_{0\leq\lambda\leq 1}\lambda H(\lambda)$$

is a fuzzy subset of X, and called the fuzzy subset determined by H.

Lemma 1[2] Let H be a nest sets on group G, then T(H) is a fuzzy subgroup of G iff $H(\lambda)$, $\lambda\in[0,1]$, is subgroup of G.

Lemma 2[3] Let T(H) be fuzzy subset determined by nest sets H on set X, then

$$T(H)_{(\lambda)}\subset H(\lambda)\subset T(H)_\lambda,$$

where, $T(H)_{(\lambda)}$ and $T(H)_\lambda$ are, respectively, strong λ-cut set and λ-cut set of T(H).

The proof of Theorem 1.

Put $H(\lambda) = \langle A_\lambda \rangle$, then

$\lambda_1 < \lambda_2 \Longrightarrow A_{\lambda_1} \supset A_{\lambda_2} \Longrightarrow \langle A_{\lambda_1} \rangle \supset \langle A_{\lambda_2} \rangle \Longrightarrow H(\lambda_1) \supset H(\lambda_2)$.

This shows that $H: \lambda \mid\rightarrow H(\lambda)$ is a nest set on group G. Also, $H(\lambda)=\langle A_\lambda \rangle$ is a subgroup of G, from lemma 1, it follows that fuzzy subset

$$T(H)=\bigcup_{0\leqslant\lambda\leqslant 1} \lambda H(\lambda)=\bigcup_{0\leqslant\lambda\leqslant 1} \lambda \langle A_\lambda \rangle$$

determined by H is a fuzzy subgroup of G. We will prove that $T(H)=\langle A \rangle$.

From lemma 2 we have $T(H)_{(\lambda)} \subset H(\lambda) \subset T(H)_\lambda$. Also,

$$A_\lambda \subset \langle A_\lambda \rangle = H(\lambda) \subset T(H)_\lambda,$$

Hence

$$A=\bigcup_{0\leqslant\lambda\leqslant 1} \lambda A_\lambda \subset \bigcup_{0\leqslant\lambda\leqslant 1} \lambda T(H)_\lambda = T(H).$$

Suppose that K is any fuzzy subgroup in G containing A, then $A_\lambda \subset K_\lambda$, $\lambda \in [0,1]$. Since K_λ is a subgroup, $\langle A_\lambda \rangle \subset K_\lambda$, and so

$$T(H)=\bigcup_{0\leqslant\lambda\leqslant 1} \lambda \langle A_\lambda \rangle \subset \bigcup_{0\leqslant\lambda\leqslant 1} \lambda K_\lambda = K.$$

From definition 1 we see that $\langle A \rangle = T(H)$. This completes the proof.

Corollary 1 Let A be a fuzzy subset of grop G. If $\mathrm{Card\,Im}A < +\infty$, then

$$\mathrm{Im}A \supset \mathrm{Im}\langle A \rangle,\ \mathrm{Card\,Im}A \geqslant \mathrm{Card\,Im}\langle A \rangle.$$

<u>Example 1</u> Let

$$G = \{ e, \alpha, \alpha^2, \alpha^3, \beta, \alpha\beta, \alpha^2\beta, \alpha^3\beta \}$$

be a Octic Group, where $\alpha^4 = e = \beta^2$, $\beta\alpha = \alpha^{-1}\beta$,

$t_i \in [0,1]$, $i=0,1,\dots,6$, $t_0 > t_1 > \dots > t_6$.

We define a fuzzy subset A of G as follows:

$e \rightarrow t_0$, $\alpha^2 \rightarrow t_1$, $\alpha \rightarrow t_2$, $\alpha^3 \rightarrow t_3$, $\beta \rightarrow t_4$, $\alpha\beta \rightarrow t_5$, $\alpha^2\beta \rightarrow t_5$, $\alpha^3\beta \rightarrow t_6$.

Determine fuzzy subgroup $\langle A \rangle$ generated by A.

Solution Clearly,

$A_{t_0}=\{e\}$, $A_{t_1}=\{e, \alpha^2\}$, $A_{t_2}=\{e, \alpha, \alpha^2\}$, $A_{t_3}=\{e, \alpha, \alpha^2, \alpha^3\}$,

$A_{t_4}=\{e, \alpha, \alpha^2, \alpha^3, \beta\}$, $A_{t_5}=\{e, \alpha, \alpha^2, \alpha^3, \beta, \alpha\beta, \alpha^2\beta\}$, $A_{t_6}=G$.

Hence,

$<A_{t_0}>=\{e\}, <A_{t_1}>=\{e, a^2\}, <A_{t_2}>=<A_{t_3}>=\{e, a, a^2, a^3\}$,

$<A_{t_4}>=<A_{t_5}>=<A_{t_6}>=G$.

From Theorem 1 we have

$$<A>(x)=\bigcup_{0\leq i\leq 1} t_i<A_{t_i}>(x)=\sup_{x\in<A_{t_i}>} t_i,\ x\in G.$$

Hence,

$$<A>(e)=t_0, <A>(a^2)=t_1, <A>(a)=<A>(a^3)=t_2,$$

$$<A>(\beta)=<A>(a\beta)=<A>(a^2\beta)=<A>(a^3\beta)=t_4.$$

Remark: This example has been discussed by N. V. Dixit in [1], however, our method is totally different from [1].

Example 2 Let G=(Z,+) be integers additive group. A fuzzy subset A of G is defined as follows:

$$A(0)=A(1)=A(-1)=0, A(k)=1-(1/|k|), k=\pm 2, \pm 3, \pm 4, \dots .$$

Determine <A>.

Solution Clear, ImA={0, 1/2, 2/3, 3/4, ...}, and

$$A_\lambda=Z,\ \lambda=0; A_\lambda=\{k: k\in Z, |k|\geq n\},\ \lambda=(n-1)/n, n=2,3,4,\dots .$$

From this we get that $<A_\lambda>=Z$ holds for each $\lambda\in ImA$. Thus

$$<A>=\bigcup_{\lambda\in ImA}\lambda<A_\lambda> = \bigcup_{\lambda\in ImA}\lambda Z.$$

This shows that

$$<A>(k) = (\bigcup_{\lambda\in ImA}\lambda Z)(k) = \sup_{\lambda\in ImA}\lambda = 1,\ k\in Z.$$

Therefore <A> is just group Z.

In this example, $1\in Im<A>$, but $1\notin ImA$.

From Example 1 and 2 we see that ImA and Im<A> are, in general, not contain each other.

In the following, we will discuss the relation between generating systems of conjugate fuzzy subgroups.

Definition 3[1] Let G be a group, and H_1 and H_2 two fuzzy subgroups of G. H_1 and H_2 are said to be conjugate, if there exists $g\in G$ such that

$H_1(x) = H_2(g^{-1}xg)$, $x \in G$.

Let H be a fuzzy subgroup of group G, $g \in G$. We define

$$H_g^*(x) = H(g^{-1}xg), \ x \in G.$$

Then H_g^* is a fuzzy subgroup of G, and it is called conjugate fuzzy subgroup determined by H and g.

If we regard g as its characteristic function χ_g, then from the multiplicative rule of fuzzy subsets in group, it follows that

$$(gAg^{-1})(x)=\sup_{uvw=x}(\chi_g(u) \wedge A(v) \wedge \chi_g(w))=A(g^{-1}xg).$$

Specially, when A=H is a fuzzy subgroup of G,

$$(gHg^{-1})(x) = H(g^{-1}xg) = H_g^*(x),$$

i.e., $gHg^{-1} = H_g^*$.

Lemma 3 Let A be a fuzzy subset of group G, $x \in G$. Then

$$(xAx^{-1})_\lambda = xA_\lambda x^{-1}, (xAx^{-1})_{\langle\lambda\rangle} = xA_{\langle\lambda\rangle}x^{-1}.$$

Proof $z \in (xAx^{-1})_\lambda \iff (xAx^{-1})(z) = A(x^{-1}zx) \geqslant \lambda$

$\iff x^{-1}zx \in A_\lambda \iff z \in xA_\lambda x^{-1}$;

$z \in (xAx^{-1})_{\langle\lambda\rangle} \iff (xAx^{-1})(z) = A(x^{-1}zx) > \lambda$

$\iff x^{-1}zx \in A_{\langle\lambda\rangle} \iff z \in xA_{\langle\lambda\rangle}x^{-1}$.

Theorem 2 Let H be a fuzzy subgroup of group G, $g \in G$. Then

$$(H_g^*)_\lambda = gH_\lambda g^{-1}, \ (H_g^*)_{\langle\lambda\rangle} = gH_{\langle\lambda\rangle}g^{-1}.$$

Proof It follows from Lemma 3.

Theorem 3 portraies the relation between the cut sets and strong cut sets of conjugate fuzzy subgroups.

Lemma 4 Let S be crisp set of group G. Then

$$\langle gSg^{-1}\rangle = g\langle S\rangle g^{-1}, \ g \in G.$$

Proof Clearly,

$$S\subset\langle S\rangle \Longrightarrow \langle gSg^{-1}\rangle\subset g\langle S\rangle g^{-1}.$$

On the other hand, for each $x\in g\langle S\rangle g^{-1}$, let

$$x=gyg^{-1},\ g\in\langle S\rangle,\ g=s_1^{k1}s_2^{k2}\ldots s_t^{kt}.$$

Then

$$x = gyg^{-1} = gs_1^{k1}s_2^{k2}\ldots s_t^{kt}g^{-1} = (gs_1g^{-1})^{k1}\ldots(gs_tg^{-1})^{kt} \in\langle gSg^{-1}\rangle.$$

Hence $g\langle S\rangle g^{-1}\subset\langle gSg^{-1}\rangle$.

Lemma 5[8] Let A be a fuzzy subset of X, and H a nest sets on X, and T(H) the fuzzy subset determined by H. If

$$A_{\langle\lambda\rangle}\subset H(\lambda)\subset A_\lambda,$$

then T(H) = A.

The following theorem portraies the relation between generating system of conjugate fuzzy groups.

Theorm 3 Let H be a fuzzy subgroup of group G, $g\in G$. Then A is fuzzy generating system of H iff gAg^{-1} is fuzzy generating system of H_g^*.

Proof Let A be generating system of H. We will prove that gAg^{-1} is generating system of H_g^*. By Theorem 1, $H = \langle A\rangle = \bigcup_{\lambda\in[0,1]}\lambda\langle A_\lambda\rangle$. From Lemma 2, $H_{\langle\lambda\rangle}\subset\langle A_\lambda\rangle\subset H_\lambda$. Then

$$gH_{\langle\lambda\rangle}g^{-1}\subset g\langle A_\lambda\rangle g^{-1}\subset gH_\lambda g^{-1}.$$

Also,

$$gH_{\langle\lambda\rangle}g^{-1} = (H_g^*)_{\langle\lambda\rangle},\quad g\langle A_\lambda\rangle g^{-1} = \langle gA_\lambda g^{-1}\rangle = \langle(gAg^{-1})_\lambda\rangle,$$

$$gH_\lambda g^{-1} = (H_g^*)_\lambda.$$

Hence

$$(H_g^*)_{\langle\lambda\rangle}\subset\langle(gAg^{-1})_\lambda\rangle\subset(H_g^*)_\lambda.$$

Also, the fuzzy subset determined by nest sets $\lambda\mapsto\langle(gAg^{-1})_\lambda\rangle$ is just $\langle gAg^{-1}\rangle = \bigcup_{0\le\lambda\le1}\lambda\langle(gAg^{-1})_\lambda\rangle$. It follows from Lemme 5 that

$H_g^* = \langle gAg^{-1} \rangle$. This shows that gAg^{-1} is fuzzy generating system of H_g^*.

Conversely, since H is conjugate fuzzy subgroup determined by H_g^* and g^{-1}, when gAg^{-1} is fuzzy generating system of H_g^*, from above inference we get $g^{-1}(gAg^{-1})g = A$ is fuzzy generating system of H. This completes the proof.

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