Fuzzy Order-Homomorphism on Groups

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Abstract: In this paper, the concept of fuzzy order-homomorphism on groups is introduced and some of its properties are studied. We prove that a fuzzy order-homomorphism on groups can be defined by an ordinary group homomorphism and a finitely meet-preserving order homomorphism. Using this result we obtain a pointwise charactrisation of fuzzy order-homomorphism on groups.

Keywords: Fuzzy lattice, order-homomorphism, fuzzy order-homomorphism on group

1. Preliminaries

Throughout this paper, L, L_1, L_2 always denote the fuzzy lattices, i.e. completely distributive lattices with order-reversing involutions $\alpha \mapsto \alpha'$. 0 and 1 are their smallest element and greatest element respectively. A fuzzy lattice L is said to be reqular[5] if the intersection of every pair of non-zero elements of L is not zero. L^X will denote the family of all L-fuzzy sets on X. For $A \in L^X$, the set $\{x \in X \mid A(x) > 0\}$ is called the support of A and denoted by *supp* A, and the value $\bigvee\{A(x) \mid x \in X\}$ is called the height of A and denoted by *hgt* A. When *supp* A is a singleton, A is called L-fuzzy point and denoted by x_λ where $x =$ *supp* A and $\lambda =$ *hgt* A. $\tilde{X}(L)$ will denote the set of all L-fuzzy points on X. For simplicity, a L-fuzzy set on X which takes the constant value λ on X (or Y) is still denoted by λ. We assume that for the empty family $\emptyset$, $\bigvee\emptyset = 0$ and $\bigwedge\emptyset = 1$.

Let X be a group. Using Zadeh's extension principle, we define the multiplicative operator of L-fuzzy sets on X as follows: for A, $B \in L^X$,

$$(A \cdot B)(x) = \bigvee_{st = x} [A(s) \wedge B(t)].$$

In paticular, for L-fuzzy points, if L is regular then

$$x_\lambda \cdot y_\mu = (x \cdot y)_{\lambda \wedge \mu}.$$

Definition 1.1[5]. A mapping $f : L_1 \to L_2$ is called an order-homomorphism, if the following conditions hold:

(H_1) $f(0) = 0$;

(H_2) f is union-preserving, i.e. $f(\bigvee a_t) = \bigvee f(a_t)$;

(H_3) f^{-1} is complement-preserving, i.e. for each $b \in L_2$, $f^{-1}(b') = [f^{-1}(b)]'$, where the mapping $f^{-1} : L_2 \to L_1$ is defined as

$$f^{-1}(b) = \bigvee\{b \in L_1 : f(a) \le b\}.$$

Obviously, if $f : L_1 \to L_2$ is an order-homomorphism, then $f^{-1}(1) = 1$ and $f^{-1}(0) = 0$ (See [2]).

Lemma 1.1. Suppose that $F : L_1^X \to L_2^Y$ is a mapping and there exist the mappings $f : X \to Y$ and $\varphi : L_1 \to L_2$ such that

$$F(x_\lambda) = [f(x)]_{\varphi(\lambda)} \in \tilde{Y}(L_2), \quad \text{for any } x_\lambda \in \tilde{X}(L_1).$$

(1) If F is union-preserving, then F is a bi-induced mapping[3] of f and φ, i.e.

$$F(A)(y) = \bigvee\{\varphi(A(x)) \mid f(x) = y\}, \quad \text{for any } A \in L_1^X \text{ and } y \in Y. \tag{1.1}$$

(2) If F is order-preserving, then

$$F^{-1}(B)(x) = \varphi^{-1}(B(f(x)) \quad \text{for any } B \in L_2^Y \text{ and } x \in X. \tag{1.2}$$

Lemma 1.2. Suppose that $F : L_1^X \to L_2^Y$ is a bi-induced mapping of the mapping $f : X \to Y$ and $\varphi : L_1 \to L_2$ satisfying $\varphi^{-1}(0) = 0$. Then

$$F(x_\lambda) = [f(x)]_{\varphi(\lambda)} \in \tilde{X}(L_1).$$

2. Fuzzy order-homomorphism on groups and its structures

Definition 2.1. Let X and Y be two groups. A mapping $F : L_1^X \to L_2^Y$ is called a fuzzy order-homomorphism on groups, if it is an order-homomorphism (i.e. fuzz function [4]) and satisfies

$$F(A \cdot B) = F(A) \cdot F(B), \quad \text{for all } A, B \in L^X. \tag{2.1}$$

Remark 2.1. Suppose that $f : X \to Y$ is a usual homomorphism of groups X to Y and $L = [0,1]$, then the function of Zadeh's type $F : L^X \to L^Y$ induced by f is a fuzzy order-homomorphism on groups.

In the following, we always assume that L_1 is regular. X and Y are two groups, e denotes the unit element in groups

Proposition 2.1. Let $F : L_1^X \to L_2^Y$ be a fuzzy order-homomorphism on groups. Then

(1) F takes fuzzy points on X to fuzzy points on Y, and

$$supp\ F(x_\lambda) = supp\ F(x_\mu), \quad \text{for any } \lambda, \mu \in L_1 - \{0\}.$$

(2) $supp\ F(e_\lambda) = e$ and $hgt\ F(e_\lambda) = hgt\ F(x_\lambda)$, for any $x \in X$ and $\lambda \in L_1 - \{0\}$,

Proof. (1) See Lemma 2.2 in [5]. (2) It follows easily from Definition 2.1.

Theorem 2.1. Suppose that $F : L_1^X \to L_2^Y$ is a fuzzy order-homomorphism on groups, then there exist an ordinary group homomorphism $f : X \to Y$ and a finitely meet-preserving order-homomorphism $\varphi : L_1 \to L_2$ such that F is a bi-induced mapping of $f$and φ, and (1.2) holds.

Proof. Define the mappings $f : X \to Y$ and $\varphi : L_1 \to L_2$ as follows, respectively:

$$f(x) = supp\ F(x_1) \quad \text{and} \quad \varphi(\lambda) = \begin{cases} hgt\ F(e_\lambda), & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0. \end{cases}$$

From Proposition 2.1, $supp\ F(x_\lambda) = supp\ F(x_1) = f(x)$ and $hgt\ F(x_\lambda) = hgt\ F(e_\lambda) = \varphi(\lambda)$, hence $F(x_\lambda) = [f(x)]_{\varphi(\lambda)}$ for any $x_\lambda \in \tilde{X}(L_1)$. Thus by Lemma 1.1, we know that F is a bi-induced mapping of f and φ, and (1.2) holds. We easily prove that f is an ordinary group homomorphism and φ is a finitely meet-preserving order-homomorphism.

Theorem 2.2. suppose that $f : X \to Y$ is an ordinary group homomorphism and $\varphi : L_1 \to L_2$ is a finitely meet-preserving order-homomorphism. Then the bi-induced mapping $F : L_1^X \to L_2^Y$ of f and φ is a fuzzy order-homomorphism on groups.

proof. Let $F : L_1^X \to L_2^Y$ be the bi-induced mapping of f and φ. By (1.1) and the union-preserving property of φ, it is easy to show that $F(0) = 0$ and F is union-preserving.

By Lemma 1.2, Lemma 1.1 and the complement-preserving property of φ^{-1}, it is easy to show that $F^{-1}(B') = [F^{-1}(B)]'$, for any $B \in L_2^Y$. Therefore F is an order-homomorphism.

Next, we prove (2.1). When $y \notin f(X)$, $[F(A\cdot B)](y) = 0 = [F(A)\cdot F(B)](y)$; When $y \in f(X)$, since φ is union-preserving and finitely meet-preserving, we have

$$F(A\cdot B)(y) = \bigvee_{f(x)=y} \varphi[(A\cdot B)(x)] = \bigvee_{f(x)=y} \bigvee_{u\cdot v=x} [\varphi(A(u)) \wedge \varphi(B(v))]. \tag{2.2}$$

and

$$[F(A)\cdot F(B)](y) = \bigvee_{s\cdot t=y} [(\bigvee_{f(u)=s} \varphi(A(u))) \wedge (\bigvee_{f(v)=t} \varphi(B(v)))]. \tag{2.3}$$

(2.1) follows easily From (2.2) and (2.3). This completes the proof.

From Theorem 2.1 and Theorem 2.2, we have the following result immediately.

Theorem 2.3. The mapping $F: L_1^X \to L_2^Y$ is a fuzzy order-homomorphism on groups if and only if there exist an ordinary group homomorphism $f: X \to Y$ and a finitely meet-preserving order-homomorphism $\varphi: L_1 \to L_2$ such that F is a bi-induced mapping of f and φ.

3. Pointwise characterisation of fuzzy order-homomorphism on groups

Definition 3.1. A mapping $\tilde{f}: \tilde{X}(L_1) \to \tilde{Y}(L_2)$ is called a L-fuzzy homomorphism on groups if the following condition holds:

$$\tilde{f}(x_\lambda \cdot y_\mu) = \tilde{f}(x_\lambda)\cdot \tilde{f}(y_\mu), \quad \text{for all } x_\lambda,\ y_\mu \in \tilde{X}(L_1).$$

In particular, when $L_1 = L_2 = [0,1]$, the L-fuzzy homomorphism on groups is briefly called the fuzzy homomorphism on groups (See [2])

Definition 3.2. A mapping $\tilde{f}: \tilde{X}(L_1) \to \tilde{Y}(L_2)$ is called a pointwise fuzzy order-homomorphism on groups if the following conditions are satisfied:

(1) $\tilde{f}$ is a L-fuzzy homomorphism on groups;

(2) $\tilde{f}(e_{\vee\lambda_t}) = \bigvee \tilde{f}(e_{\lambda_t})$;

(3) $hgt\ \tilde{f}^{-1}(e_{\lambda'}) = [hgt\ \tilde{f}^{-1}(e_\lambda)]'$, for $\lambda \in L_2 - \{0,1\}$.

Proposion 3.1. Suppose that $\tilde{f}: \tilde{X}(L_1) \to \tilde{Y}(L_2)$ is a pointwise fuzzy order-homomorphism on groups. Then

(1) $supp\ \tilde{f}(x_\lambda) = supp\ \tilde{f}(x_\mu)$, for any $\lambda, \mu \in L_1 - \{0\}$;

(2) $supp\ \tilde{f}(e_\lambda) = e$ and $hgt\ \tilde{f}(e_\lambda) = hgt\ \tilde{f}(x_\lambda)$, for any $x_\lambda \in \tilde{X}(L_1)$.

The proof is similar to that of Proposion 2.1.

Lemma 3.1. Suppose that $\tilde{f}: \tilde{X}(L_1) \to \tilde{Y}(L_2)$ is a mapping which satisfies: there exist the mappings $f: X \to Y$ satisfying $f^{-1}(e) \neq \emptyset$ and $\varphi: L_1 \to L_2$ such that $\tilde{f}(x_\lambda) = [f(x)]_{\varphi(\lambda)}$, for any $x_\lambda \in \tilde{X}(L_1)$. Then

$$hgt\ \tilde{f}^{-1}(e_\mu) = \varphi^{-1}(\mu), \quad \text{for any } \mu \in L_2 - \{0\}. \tag{3.1}$$

Proof. It follows from (1.1) and the fact $\tilde{f}(x_\lambda) = [f(x)]_{\varphi(\lambda)}$.

Theorem 3.1 $\tilde{f} : \tilde{X}(L_1) \to \tilde{Y}(L_2)$ is a pointwise fuzzy order-homomorphism on groups if and only if there exist an ordinary gpoup homomorphism $f : X \to Y$ and a finitely meet-preserving order-homomorphism $\varphi : L_1 \to L_2$ such that

$$\tilde{f}(x_\lambda) = [f(x)]_{\varphi(\lambda)}, \quad \text{for any } x_\lambda \in \tilde{X}(L_1). \tag{3.2}$$

Proof. Necessity. Define the mappings $f : X \to Y$ and $\varphi : L_1 \to L_2$ as follows, respectively:

$$f(x) = supp\ \tilde{f}(x_1), \quad \text{and} \quad \varphi(\lambda) = hgt\ \tilde{f}(e_\lambda),\ \varphi(0) = 0.$$

From Proposition 3.1, it follows that $supp\ \tilde{f}(x_\lambda) = supp\ \tilde{f}(x_1) = f(x)$, and $hgt\ \tilde{f}(x_\lambda) = hgt\ \tilde{f}(e_\lambda) = \varphi(\lambda)$. Hence (3.2) holds. Thus by Definition 3.2 and Lemma 3.1 we easily varify that f is a ordinary group homomorphism and φ is a finitely meet-preserving order-homomorphism.

Sufficiency. By the assumption of theorem and Lemma 3.1, it is easy to verify that $\tilde{f}$ satisfies (1)-(3) of Definition 3.2. Therefore, $\tilde{f}$ is a pointwise fuzzy order-homomorphism on groups.

Theorem 3.2 Suppose that $F : L_1^X \to L_2^Y$ is a fuzzy order-homomorphism on groups. Then the restriction $\tilde{f} = F|_{\tilde{X}(L_1)}$ of F on $\tilde{X}(L_1)$ is a pointwise fuzzy order-homomorphism on groups. Conversely, suppose that $\tilde{f} : \tilde{X}(L_1) \to \tilde{Y}(L_2)$ is a pointwise fuzzy order-homomorphism on groups. Then there exists a unique fuzzy order-homomorphism on groups $F : L_1^X \to L_2^Y$ such that $\tilde{f} = F|_{\tilde{X}(L_1)}$.

Proof. Suppose that F is a fuzzy order-homomorphism on groups. By Theorem 2.1 we know that there exist an ordinary group homomorphism $f : X \to Y$ and a finitely meet-preserving order-homomorphism $\varphi : L_1 \to L_2$ such that F is a bi-induced mapping of f and φ. From Lemma 1.2 it follows that $F(x_\lambda) = [f(x)]_{\varphi(\lambda)}$. Noting that $\tilde{f} = F|_{\tilde{X}(L_1)}$, and so $\tilde{f}(x_\lambda) = [f(x)]_{\varphi(\lambda)}$. Thus by Theorem 3.1 we know that $\tilde{f} = F|_{\tilde{X}(L_1)}$ is a pointwise fuzzy order-homomorphism on groups.

Conversely, let $\tilde{f} : \tilde{X}(L_1) \to \tilde{Y}(L_2)$ be a pointwise fuzzy order-homomorphism on groups. By Theorem 3.1 there exist an ordinary group homomorphism $f : X \to Y$ and a finitely meet-preserving order-homormorphism $\varphi : L_1 \to L_2$ such that $\tilde{f}(x_\lambda) = [f(x)]_{\varphi(\lambda)}$ for $x_\lambda \in \tilde{X}(L_1)$. Letting the mapping $F : L_1^X \to L_2^Y$ be a bi-induced mapping of f and φ. From Theorem 2.3 we know that F is a fuzzy order-homomorphism on groups. By Lemma 1.2 we have $F(x_\lambda) = [f(x)]_{\varphi(\lambda)}$, and so $\tilde{f} = F|_{\tilde{X}(L_1)}$. The proof of uniqueness is straightforward.

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