FURTHER DISCUSSIONS ON GENERALIZED FUZZY INTEGRALS ON FUZZY SETS (I): On generalized convergence theorems

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Abstract

In the series papers, we will further investigate generalized fuzzy integrals on fuzzy sets given by Wu[4]. This paper is discussing the convergence of sequences of generalized fuzzy integrals on fuzzy sets which include the sequences of fuzzy measures on fuzzy sets. Various kinds of generalized convergence theorems are obtained.

Keywords: Fuzzy measure on fuzzy sets, Generalized fuzzy integral on fuzzy sets.

1. Introduction

Since Sugeno[2] brought out the concepts of fuzzy measures and fuzzy integrals, the theory has been made deeper by Ralescu and Adams[1], Wang[3], and many others. It is worth reminding that the generalized fuzzy integral on fuzzy sets (in short, (G) fuzzy integral on fuzzy sets) introduced by Wu[7] is a much wider one. In ref [6, 7], we have established convergence theorems of fuzzy integrals with Sugeno's sense or Ralescu's sense. In this paper, we will extend the convergence theorems of (G) fuzzy integrals on fuzzy sets, by using the theory of convergence of fuzzy measures given in [6, 7]. It can be viewed as an extension of what in [6, 7].

2. Fuzzy measures and generalized fuzzy integrals on fuzzy sets

Let X be a fixed set, P(X) is the power set of X, $\tilde{\mathscr{A}}$ is a fuzzy σ-algebra formed by the fuzzy subset of $\underset{\sim}{X}$. Let $\widetilde{F}(X)$ denote the set of all $\mathscr{A}$ -measurable functions from X to $[0, +\infty]$.

Definition 2. 1[1]. A set-function $\mu: \mathscr{A} \to [0, +\infty]$ is said to be a fuzzy measure on fuzzy sets if it satisfies the following conditions:

(F_1) $\mu(\varphi)=0$;

(F_2) $\widetilde{A} \subset \widetilde{B}$ implies $\mu(\widetilde{A}) \leqslant \mu(\widetilde{B})$;

(F_3) $\widetilde{A}_n \uparrow \widetilde{A}$ implies $\mu(\widetilde{A}_n) \uparrow (\widetilde{A})$;

(F_4) $\widetilde{A}_n \downarrow \widetilde{A}$, there exists a n_0, s. t. $\mu(\widetilde{A}_{n_0}) < +\infty$ implies $\mu(\widetilde{A}_n) \downarrow \mu(\widetilde{A})$, The triplet $(X, \mathscr{A}, \mu)$ is called a fuzzy measure space.

Let $\widetilde{M}(X)$ denote the set of all fuzzy measures on $(X, \tilde{\mathscr{A}})$.

Lemma 2.1[6]. Let $\mu_1, \mu_2 \in \widetilde{M}(X)$, $\tilde{A} \in \tilde{\mathscr{A}}$. If we define $\mu_1 \vee \mu_2$, $\mu_1 \wedge \mu_2$ by $(\mu_1 \vee \mu_2)(\tilde{A}) =: \mu_1(\tilde{A}) \vee \mu_2(\tilde{A})$, $(\mu_1 \wedge \mu_2)(A) =: \mu(\tilde{A}) \wedge \mu_2(\tilde{A})$, then $\mu_1 \vee \mu_2$, $\mu_1 \wedge \mu_2 \in \widetilde{M}(X)$.

Definition 2.2[6]. Let $\{\mu_n\} \subset \widetilde{M}(X)$. We define the inferior limit and the superior limit, respectively as follows:

$(\liminf\limits_{n\to\infty} \mu_n)(\tilde{A}) =: \liminf\limits_{n\to\infty} \mu_n(\tilde{A})$,

$(\limsup\limits_{n\to\infty} \mu_n)(\tilde{A}) =: \limsup\limits_{n\to\infty} \mu_n(\tilde{A})$,

where $\tilde{A} \in \tilde{\mathscr{A}}$. If there exists a set-function $\mu: \tilde{\mathscr{A}} \to [0, +\infty]$ s. t. $\mu(\tilde{A}) = (\liminf\limits_{n\to\infty} \mu_n)(\tilde{A}) = \liminf\limits_{n\to\infty} \mu_n(\tilde{A})$ for each $A \in \tilde{\mathscr{A}}$ (resp. uniform with $\tilde{A} \in \tilde{\mathscr{A}}$), then we say that $\{\mu_n\}$ converges (resp. uniformly) to μ, simply written as $\lim\limits_{n\to\infty} \mu_n = \mu$ (resp. $\mu_n \xrightarrow{U} \mu$).

Lemma 2.2[6]. Let $\{\mu_n\} \subset \widetilde{M}(X)$, and μ be a set-function from $\tilde{\mathscr{A}}$ to $[0, +\infty]$. Then

(i) $\mu_n \xrightarrow{U} \mu$ implies $\mu \in \widetilde{M}(X)$,

(ii) $\mu_n \to \mu$ does not imply $\mu \in \widetilde{M}(X)$.

Proposition 2.1. Let $\{\mu_n\} \subset M(X)$, and μ be a set-function from $\tilde{\mathscr{A}}$ to $[0, +\infty]$. If each $\mu_n (n \geqslant 1)$ is null-additive (resp. subadditive, superadditive,)[3], then $\mu_n \to \mu$ implies μ is null-additive (resp, subadditive, superadditive).

Proposition 2.2. Let $\{\mu_n\} \subset \widetilde{M}(X)$, and μ be a set-function from $\tilde{\mathscr{A}}$ to $[0, +\infty]$. If each $\mu_n (n \geqslant 1)$ is autocontinuous (resp. from above, from below)[3], then

(i) $\mu_n \xrightarrow{U} \mu$ implies μ is autocontinuous (resp. from above, from below);

(ii) $\mu_n \to \mu$ does not implies μ is autocontinuous (resp. from above, from below).

Definition 2.3. Let $D = [0, +\infty] \times [0, +\infty] \setminus \{(0, +\infty), (+\infty, 0)\}$.

If $S: D \to [0, +\infty]$ satisfies the following conditions:

(i) $S(0, x) = 0$, for each $x \in (0, +\infty)$, and there exists a $e \in (0, +\infty]$, s. t. $S(x, e) = x$, for each $x \in (0, +\infty]$, e is called identity of S;

(ii) $a \leqslant b$, $c \leqslant d$ implies $S(a, c) \leqslant S(b, d)$;

(iii) $S(a, b) = S(b, a)$;

(iv) $\{(x_n, y_n)\} \subset D$, $(x, y) \in D$, if $x_n \uparrow x$, $y_n \downarrow y$, then $S(x_n, y_n) \to S(x, y)$,

then S is called a generalized triangle norm.

Definition 2.4. Let $\mu \in \widetilde{M}(X)$, $f \in \widetilde{F}(X)$, $\tilde{A} \in \tilde{\mathscr{A}}$. Then the (G) fuzzy integral on $\tilde{A}$ is defined as

$$\int_{\tilde{A}} f d\mu =: \bigvee_{\alpha > 0} S(\alpha, \mu(\tilde{A} \cap \chi_{F_\alpha}))$$

where $F_\alpha = \{x \in X: f(x) \geqslant \alpha\}$, χ_{F_α} is the characteristic function of F_α.

Lemma 2. 3. (Transformation theorem of fuzzy integral).
Let $f\in\widetilde{F}(x)$, $\widetilde{A}\in\widetilde{\mathscr{A}}$, $\mu\in\widetilde{M}(X)$. Then

$$\int_{\widetilde{A}} f d\mu = \int_0^{+\infty} \mu(\widetilde{A}\cap X_{F_a}) d\alpha \qquad (2.1)$$

where α is the Lebesgue measure on $[0, +\infty]$, and the right-hand integral $\int_0^{+\infty}\mu(A\cap X_{F_a})d\alpha$ is also a (G) fuzzy integral on fuzzy sets.

Remark. If $\mu(\widetilde{A})=M<+\infty$, Eq (2. 1) can also be as

$$\int_{\widetilde{A}} f d\mu = \int_0^{M} \mu(\widetilde{A}\cap X_{F_a}) d\alpha \qquad (2.2)$$

3. Generalized convergence theorems

Proposition 3. 1. Let $\{\mu_n(n\geqslant 1), \mu\}\subset\widetilde{M}(X)$, $f\in\widetilde{F}(X)$, $\widetilde{A}\in\widetilde{\mathscr{A}}$. Then

(i) $\mu_n\uparrow\mu$ (i. e. $\mu_1(\widetilde{A})\leqslant\mu_2(\widetilde{A})\cdots$ for each $\widetilde{A}\in\widetilde{\mathscr{A}}$, and $\mu_n\to\mu$) implies $\int_{\widetilde{A}} f d\mu_n\uparrow\int_{\widetilde{A}} f d\mu$;

(ii) $\mu_n\downarrow\mu$, and there exists a n_0, s. t. $\mu_{n_0}(\widetilde{A})<+\infty$, implies $\int_{\widetilde{A}} f d\mu_n\downarrow\int_{\widetilde{A}} f d\mu$

Proposition 3. 2. Let $\{\mu_n\}\subset\widetilde{M}(X)$, $f\in\widetilde{F}(X)$, $\widetilde{A}\in\widetilde{\mathscr{A}}$. Then

(i) $\liminf_{n\to\infty}\mu_n\in\widetilde{M}(X)$ implies $\int_{\widetilde{A}} f d(\liminf_{n\to\infty}\mu_n)\leqslant\liminf_{n\to\infty}\int_{\widetilde{A}} f d\mu_n$,

(ii) $\limsup_{n\to\infty}\mu_n\in\widetilde{M}(X)$ implies $\limsup_{n\to\infty}\int_{\widetilde{A}} f\mu_n\leqslant\int_{\widetilde{A}} f d(\limsup_{n\to\infty}\mu_n)$.

Theorem 3. 1 (Generalized monotone convergence theorem).
Let $\{\mu_n(n\geqslant 1), \mu\}\subset\widetilde{M}(X)$, $\{f_n(n\geqslant 1), f\}\subset\widetilde{F}(X)$, $\widetilde{A}\in\widetilde{\mathscr{A}}$. Then

(i) $f_n\uparrow f$, $\mu_n\uparrow\mu$ implies $\int_{\widetilde{A}} f_n d\mu_n\uparrow\int_{\widetilde{A}} f d\mu$,

(ii) $f_n\downarrow f$, $\mu_n\downarrow\mu$, and there exists a n_0, s. t. $\mu_{n_0}(\widetilde{A})<+\infty$, implies $\int_{\widetilde{A}} f_n d\mu_n\downarrow\int_{\widetilde{A}} f d\mu$.

Theorem 3. 2 (Generalized Fatou's lemmas). Let $\{\mu_n\}\subset\widetilde{M}(X)$, $\{f_n\}\subset\widetilde{F}(X)$, and $\{\bigwedge_{k=n}^{\infty}\mu_k\}\subset\widetilde{M}(X)$, $\{\bigvee_{k=n}^{\infty}\mu_k\}\subset\widetilde{M}(X)$, $\widetilde{A}\in\widetilde{\mathscr{A}}$.

(i) If $\liminf_{n\to\infty}\mu_n\in\widetilde{M}(X)$, then

$$\int_{\widetilde{A}}(\liminf_{n\to\infty} f_n)\, d\,(\liminf_{n\to\infty}\mu_n)\leqslant\liminf_{n\to\infty}\int_{\widetilde{A}} f_n d\mu_n;$$

(ii) If $\limsup_{n\to\infty} \mu_n \in \widetilde{M}(X)$, and there exists a n_0 s.t. $\mu_{n_0}(\widetilde{A}) < +\infty$, then

$$\limsup_{n\to\infty} \int_X f_n d\mu_n \leqslant \int_X (\limsup_{n\to\infty} f_n)\, d\,(\limsup_{n\to\infty} \mu_n).$$

Theorem 3.3 (Generalized Lebesgue convergence theorem). Let $\{\mu_n(n\geqslant 1), \mu\} \subset \widetilde{M}(X)$, $\{f_n(n\geqslant 1), f\} \subset \widetilde{F}(X)$, $\widetilde{A} \in \widetilde{\mathscr{A}}$. If $\{\bigvee_{k=n}^{\infty} \mu_k\}, \{\bigwedge_{k=n}^{\infty} \mu_k\} \subset \widetilde{M}(X)$, and there exists a n_0, s.t. $\mu_{n_0}(\widetilde{A}) < +\infty$, then $f_n \to f$, $\mu_n \to \mu$ implies

$$\int_X f_n d\mu_n \to \int_X f d\mu.$$

Concluding remarks:

In a subsequent paper, we will show the generalized fuzzy integrals of fuzzy-valued functions on fuzzy sets.

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