THE PARETO OPTIMUM FOR FUZZY MULTIPLE OBJECTIVE PROGRAMMING

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ABSTRACT

In this paper, the multiple objective linear programming problem with imprecise objective and constraint coefficients is studied based on the fuzzy set theory. Using the method of ranking fuzzy number with total integral valves, we propose an auxiliarg multiple objective linear programming model to resolve the imprecise nature. Further, We develop an extended Zimmermann's approach to solve theauxiliarg multiple objective linear programming problem, and the α–Pareto optimum solution of the original fuzzy multiple objective linear programming is derived.

Keywords: fuzzy programming, multiple objective programing, Pareto optimum.

1. Introduction

In practice, there are many multiple objective programming(MOP)problem that cannot be modeled in a classical way because the different elements of the problem are vaguely defined. Thisimprecise nature has long been studiedwith the help of probability theory.However, probability theory might not give us correct meaning to solve some practial decision making problems. In addition, applying probability theory to some optimization problems has negative effect on the computational efficiency. Zadeh's fuzzy sets theory appearrs to be an ideal approach to make the problem more realistic and human–consistent and hence more applicable. Thus, fuzzy multiple objective programing (FMOP) is a tool to deal with this fuzziness which causes difficulties in modeling [1, 3, 6,9, 10]. A survey on approaches, problems and methods of FMOP can be found in [3].

In this paper, we will study the multiple objective linear programming problem with imprecise objectives and constraints coefficientts which have trapezoidal membershipfunctions.In order to propose an crisp auxiliarg multiple objective linear programing model to resolve this fuzzy nature, we will first investigate the properties of trapezoidal fuzzy numbers and the method of ranking fuzzy numbers with total integral value in section 2.Then, in section 3, an auxiliary multiple objectivwe linear programming model is derived, and the concept of α–Pareto optimum solution is introduced. Last, an extended version of Zimmermann's approach is developed to solve the auxiliary multiple objective linear programming problem and the α–Pareto optimum solution is derived in the section 4.

2. Fuzzy Numbers And Ordering With Total Integral Value

The concept of fuzzy number is introduced in [2, 8].

Definition 2.1 A fuzzy number $\tilde{A}$ is a fuzzy subset of the real line R with membership function $\mu_{\tilde{A}}(x)$ which posseses the following properties:

(1) the α–cut $\tilde{A} = \{x \in R : \mu_{\tilde{A}}(x) \geqslant \alpha\}$ is a closed interval for $\forall \alpha \in (0, 1)$.

(2) supp $\tilde{A} = cl\{x \in R : \mu_{\tilde{A}}(x) > 0\}$–is also a closed interval.

(3) there exists $x \in R$ such that $\mu_{\tilde{A}}(x) = 1$

and we will denote the set of all fuzzy numbers for F(R).

Proposetion 2.1[8] A fuzzy subset $\tilde{A}$ of R is a fuzzy number if and only if its membership function $\mu_{\tilde{A}}(x)$ can be denoted by

$$\mu_{\tilde{A}}(x) = \begin{cases} L(x), & if \;\; x < m \\ 1 \;, & if \;\; m \leqslant x \leqslant n \\ R(x), & if \;\; x > n \end{cases} \tag{2.1}$$

Where L(x) is a continuous, strictly increasing function for $x < m$ and there exists m_1 m such that $L(x) = 0$ for $x \leqslant m_1$, R(x) is continuous, strictly descreasing functon for $x > n$ and there exists $n_1 \geqslant n$ such that $R(x) = 0$ for $x \geqslant n_1$. Symbolically, $\tilde{A} = ([m,n]; L(x), R(x))$ and L(X), R(x) are called the left reference function and right reference function, respectively.

For decision making in a fuzzy environment. a very important procedure is to rank fuzzy numbers . Many method of ranking fuzzy number have been developed by researchers. In this study, we employ Liou and Wang's approach of ranking fuzzy numbers with total integral value[7]. By their method, for any fuzzy numbers $\tilde{A}$ and $\tilde{B}$, $\tilde{A} \, \textcircled{T} \, \tilde{B}$ if and only if $I_T^{(\alpha)}(\tilde{A}) \, \textcircled{T} \, I_T^{(\alpha)}(\tilde{B})$. The symbol $\textcircled{T}$ denotes the operators $\leqslant$, $=$, or $\geqslant$, $I_T^{(\alpha)}(\tilde{A})$ is the total integral value of fuzzy number $\tilde{A}$ defined via

$$I_T^{(\alpha)}(\tilde{A}) = \alpha I_R(\tilde{A}) + (1-\alpha) I_L(\tilde{A}) \tag{2.2}$$

Where $I_L(\tilde{A})$, $I_R(\tilde{A})$ are the left integral value and right integral value of $\tilde{A}$ defined as

$$I_L(\tilde{A}) = \int_0^1 L^{-1}(y)dy \qquad I_R(\tilde{A}) = \int_0^1 R^{-1}(y)dy \tag{2.3}$$

respectively. The parameter $\alpha \in [0, 1]$ is the index of optimism which represents the degree of optimism of a decision maker. A larger α indicates a higher degree of optimism.

There are many different classes of fuzzy numbers, but in practice it is convenient to describe a fuzzy $\tilde{A}$ by the parametric functions

$$L(x) = max\left(\frac{x-a_1}{a_2-a_1}, 0\right) \quad R(x) = max\left(\frac{a_4-x}{a_4-a_3}, 0\right) \tag{2.4}$$

Let $\tilde{A} = ([a_2, a_3], L(x), R(x))$ and call it trapezoidal fuzzy number. A trapezoidal fuzzy number $\tilde{A}$ can be also denoted by a quadruplet $(a_1, a_2; a_3, a_4)$, and its membership function is defined as

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & if\, x < a_1 \;\; or \;\; x > a_4 \\ \dfrac{x-a_1}{a_2-a_1} & if\, a_1 \leqslant x < a_2 \\ 1 & if\, a_2 \leqslant x \leqslant a_3 \\ \dfrac{a_4-x}{a_4-a_3} & if\, a_3 < x \leqslant a_4 \end{cases} \tag{2.5}$$

and we denote the set of all trapezoidal fuzzy numbers for TF(R).

Note that if $a_2 = a_3$, the $\tilde{A} = (a_1, a_2; a_3, a_4)$ is reduced to a triangular fuzzy number. If $a_1 = a_2 = a_3 = a_4$, the $\tilde{A} = (a_1, a_2; a_3, a_4)$ is reduced to a real number.

For a trapezoidal fuzzy number $\tilde{A} = (a_1, a_2; a_3, a_4) \in TF(R)$, it can be calculated via Eq.(2.3) that

$$I_T^{(\alpha)}(\tilde{A}) = \frac{1}{2}[\alpha(a_3 + a_4) + (1-\alpha)(a_1 + a_2)] \tag{2.6}$$

for $\forall\, \alpha \in [0, 1]$. Also we can prove that

$$I_T^{(\alpha)}(k_1\tilde{A}) + k_2\tilde{B}) = k_1 I_T^{(\alpha)}(\tilde{A}) + k_2 I_T^{(\alpha)}(\tilde{B}) \tag{2.7}$$

for $\forall\; \tilde{A}, \tilde{B} \in TF(R)$ and $k_1, k_2 \geqslant 0$.

3. The Derived Crisp Model Of FMOLP

Consider the following FMOLP problem with trapezoidal fuzzy number parameters:

$$\begin{cases} \text{Max } \tilde{F}(X) = [\tilde{f}_1(X),\ \tilde{f}_2(X),\ \ldots\ \tilde{f}_p(X)]^T \\ \text{Min } \tilde{G}(X) = [\tilde{g}_1(X),\ \tilde{g}_2(X),\ \ldots\ \tilde{g}_q(X)]^T \\ \quad s.t.\quad X \in \tilde{D} = \{\ X\colon\ \tilde{A}X \textcircled{T} \tilde{\beta},\quad X \geqslant 0\ \} \end{cases} \tag{3.1}$$

Where

$$X = [x_1, x_2, \ldots x_n]^T \in R^n\ ,\quad \tilde{f}_t(X) = \sum_{k=1}^{n} \tilde{c}^{(t)} x_k,\quad \tilde{g}_s(X) = \sum_{k=1}^{n} \tilde{d}^{(s)} x_k$$

$$\tilde{A} = (\tilde{a}_{ij})_{m\times n} \in [TF(R)]^{m\times n},\quad \tilde{\beta} = [\tilde{b}_1, \tilde{b}_2, \ldots \tilde{b}_m]^T \in [TF(R)]^m$$

and $\tilde{c}^{(t)}$, $\tilde{d}^{(s)}$, $\tilde{a}_{ij}$, $\tilde{b}_i \in TF(R)$. The symbol $\textcircled{T}$ denotes the operators $\leqslant$, $=$. or $\geqslant$.

Symbolically, let

$$I_T^{(\alpha)}(\tilde{F}(X)) = [I_T^{(\alpha)}(\tilde{f}_1(X),\quad I_T^{(\alpha)}(\tilde{f}_2(X),\ \ldots\ I_T^{(\alpha)}(\tilde{f}_p(X)]^T$$

$$I_T^{(\alpha)}(\tilde{G}(X)) = [I_T^{(\alpha)}(\tilde{g}_1(X),\quad I_T^{(\alpha)}(\tilde{g}_2(X),\ \ldots\ I_T^{(\alpha)}(\tilde{g}_q(X)]^T$$

$$I_T^{(\alpha)}(\tilde{A}) = (I_T^{(\alpha)}(\tilde{a}_{ij}))_{m\times n} \in R^{m\times n}$$

$$I_T^{(\alpha)}(\tilde{\beta}) = [I_T^{(\alpha)}(\tilde{b}_1),\quad I_T^{(\alpha)}(\tilde{b}_2),\ \ldots\ I_T^{(\alpha)}(\tilde{b}_m)]^T \in R^m$$

$$I_T^{(\alpha)}(\tilde{D}) = \{\ X \in R^n:\ \ I_T^{(\alpha)}(\tilde{A})X \textcircled{T} I_T^{(\alpha)}(\tilde{\beta}),\quad X \geqslant 0\ \}$$

Thus, using Liou and Wang's method of ranking fuzzy number with integral value [7], we can derive the following auxiliary crisp MOLP model from the model (3.1):

$$\begin{cases} \text{Max } I_T^{(\alpha)}(\tilde{F}(X)) = [I_T^{(\alpha)}(\tilde{f}_1(X)),\quad I_T^{(\alpha)}(\tilde{f}_2(X)),\ \ldots\ I_T^{(\alpha)}(\tilde{f}_p(X))]^T \\ \text{Min } I_T^{(\alpha)}\tilde{G}(X) = [I_T^{(\alpha)}(\tilde{g}_1(X)),\quad I_T^{(\alpha)}(\tilde{g}_2(X)),\ \ldots\ I_T^{(\alpha)}(\tilde{g}_q(X))]^T \\ \quad s.t.\quad X \in I_T^{(\alpha)}(\tilde{D}) = \{X\colon\ \ I_T^{(\alpha)}(\tilde{A})X \textcircled{T} I_T^{(\alpha)}(\tilde{\beta}),\quad X \geqslant 0\} \end{cases} \tag{3.2}$$

The model (3.2) takes into consideration the decision maker's degree of optimism by the index $\alpha \in [0, 1]$.

For a given $\alpha \in [0, 1]$, the index of optimism, $X \in R^n$ ia calleda α–feasible solution of FMOLP (3.1), if $X \in I_T^{(\alpha)}(\tilde{D})$.A α–feasible solution $\hat{X} \in I_T^{(\alpha)}(\tilde{D})$ is said to be a α–Pareto solution (α–nondominated solution or α–efficient solution)of FMOLP (3.1), if there is no ther α–feasible solution $X \in I_T^{(\alpha)}(\tilde{D})$ such that

$$\begin{cases} I_T^{(\alpha)}(\tilde{f}_i(X)) \geqslant I_T^{(\alpha)}(\tilde{f}_i(\hat{X})) \\ I_T^{(\alpha)}(\tilde{g}_j(X)) \leqslant I_T^{(\alpha)}(\tilde{g}_j(\hat{X})) \end{cases}$$

for all i, j and

$$\begin{cases} I_T^{(\alpha)}(\tilde{f}_{i_0}(X)) > I_T^{(\alpha)}(\tilde{f}_{i_0}(\hat{X})) \\ I_T^{(\alpha)}(\tilde{g}_{j_0}(X)) < I_T^{(\alpha)}(\tilde{g}_{j_0}(\hat{X})) \end{cases}$$

for at least one i_0, j_0.

4. Two–Phase Approach To The Pareto Solution

If the index of optimism, $\alpha \in [0, 1]$, is given, we derive a crisp MOLPmodel (3.2) from FMOLP model (3.1). To solve model (3.2), we may useany MOLP technique [4] such as utility theory, goal programming orinteractive approaches. However, in this study, we suggust using theextended version of Zimmermann's fuzzy approach[5, 6, 11] toget the α–Pareto optimum solution of model (3.1).

For given $\alpha \in [0, 1]$, by solving single–objective linear programming problem, we obtain the Ideal Point $(\bar{\tilde{f}}_1^{(\alpha)}, \bar{\tilde{f}}_2^{(\alpha)}, \ldots \bar{\tilde{f}}_p^{(\alpha)}; \bar{g}_1^{(\alpha)}, \bar{g}_2^{(\alpha)}, \ldots \bar{g}_q^{(\alpha)})$ and th Anti–ideal Point $(\hat{\tilde{f}}_1^{(\alpha)}, \hat{\tilde{f}}_2^{(\alpha)}, \ldots \hat{\tilde{f}}_p^{(\alpha)}; \hat{g}_1^{(\alpha)}, \hat{g}_2^{(\alpha)}, \ldots \hat{g}_q^{(\alpha)})$. Where

$$\bar{\tilde{f}}_i^{(\alpha)} = \operatorname*{Max}_{X \in I_T^{(\alpha)}(\tilde{D})} I_T^{(\alpha)}(\tilde{f}_i(X)), \qquad \hat{\tilde{f}}_i^{(\alpha)} = \operatorname*{Min}_{X \in I_T^{(\alpha)}(\tilde{D})} I_T^{(\alpha)}(\tilde{f}_i(X))$$

$$\bar{g}_j^{(\alpha)} = \operatorname*{Min}_{X \in I_T^{(\alpha)}(\tilde{D})} I_T^{(\alpha)}(\tilde{g}_j(X)), \qquad \hat{g}_j^{(\alpha)} = \operatorname*{Max}_{X \in I_T^{(\alpha)}(\tilde{D})} I_T^{(\alpha)}(\tilde{g}_j(X))$$

For the objective function $I_T^{(\alpha)}(\tilde{f}_i(X))$ and $I_T^{(\alpha)}(\tilde{g}_j(X))$, we define the membership functions of fuzzy sets $\mu_i(I_T^{(\alpha)}(\tilde{f}_i(X)))$ and $\nu_j(I_T^{(\alpha)}(\tilde{g}_j(X)))$ by, respectively

$$\mu_i(I_T^{(\alpha)}(\tilde{f}_i(X))) = \begin{cases} 0 & if\ I_T^{(\alpha)}(\tilde{f}_i(X)) \leqslant \tilde{f}_i^{(\alpha)} \\ \dfrac{I_T^{(\alpha)}(\tilde{f}_i(X)) - f_i^{(\alpha)}}{\tilde{f}_i^{(\alpha)} - \tilde{f}_i^{(\alpha)}} & if\ \tilde{f}_i^{(\alpha)} \leqslant I_T^{(\alpha)}(\tilde{f}_i(X)) \leqslant \tilde{f}_i^{(\alpha)} \\ 1 & if\ I_T^{(\alpha)}(\tilde{f}_i(X)) \geqslant \tilde{f}_i^{(\alpha)} \end{cases}$$

$$\nu_j(I_T^{(\alpha)}(\tilde{g}_j(X))) = \begin{cases} 0 & if\ I_T^{(\alpha)}(\tilde{g}_j(X)) \geqslant \hat{g}_j^{(\alpha)} \\ \dfrac{I_T^{(\alpha)}(\tilde{g}_j(X)) - g_j^{(\alpha)}}{\bar{g}_j^{(\alpha)} - \hat{g}_j^{(\alpha)}} & if\ \bar{g}_j^{(\alpha)} \leqslant I_T^{(\alpha)}(\tilde{g}_j(X)) \leqslant \hat{g}_j^{(\alpha)} \\ 1 & if\ I_T^{(\alpha)}(\tilde{g}_j(X)) < \bar{g}_j^{(\alpha)} \end{cases}$$

Then, we solve Zimmermann's following equivalent single–objective linear programming model:

$$\begin{cases} Max\ \ \lambda^{(\alpha)} \\ s.t.\ \ \mu_i(I_T^{(\alpha)}(\tilde{f}_i(X))) \geqslant \lambda^{(\alpha)} \quad (1 \leqslant i \leqslant p) \\ \qquad \nu_j(I_T^{(\alpha)}(\tilde{g}_j(X))) \geqslant \lambda^{(\alpha)} \quad (1 \leqslant j \leqslant q) \\ \qquad X \in I_T^{(\alpha)}(\tilde{D}) \end{cases} \tag{4.1}$$

Where the extra variable $\lambda^{(\alpha)}$ is introduced by the aggregation operator 'min':

$$\lambda^{(\alpha)} = \underset{\substack{1\leqslant i\leqslant p \\ 1\leqslant j\leqslant q}}{Min}\left\{\mu_i(I_T^{(\alpha)}(\tilde{f}_i(X))),\ \ \nu_j(I_T^{(\alpha)}(\tilde{g}_i(X)))\right\}$$

Assume that the solution of model (4.1) is $(\lambda_0^{(\alpha)},\ X_0^{(\alpha)})$, where $\lambda_0^{(\alpha)} \in [0,\ 1]$ denotes the degree of compromise to which the solution $X_0^{(\alpha)}$ satisfies all of the objectives at the index of optimism α.

The result obtained by the operator 'Min' represent the worst situation and cannot be compensated by other members which may be very good. Due to this non–compensatory nature there may be multiple solution X's which will end up with the same $\lambda_0^{(\alpha)}$. That is, a solution selected randomly might be dominated by another solution with the same satisfaction level $\lambda_0^{(\alpha)}$, and the solution of (4.1) might not be unique nor efficient, which is not desirable.

To convercome the above mentioned disadvantage, Li and Lee [5, 6]propose a two–phase fuzzy approach. Zimmermann's fuzzy approach described above is used as

Phase I in which a solution $(\lambda_0^{(\alpha)}, X_0^{(\alpha)})$ is obtained. Then, a fully compensatory aggregation operator 'weighted average' is used in the second phase, and a further constraint by using $\lambda_0^{(\alpha)}$ which was obtained in Phase I is added. Thus, in Phase II, we solve the problem:

$$\begin{cases} Max \quad \lambda^{(\alpha)} = \sum_{k=1}^{p+q} w_k \lambda_k^{(\alpha)} \\ s.t. \quad \mu_i(I_T^{(\alpha)}(\tilde{f}_i(X))) \geqslant \lambda_i^{(\alpha)} \geqslant \lambda_0^{(\alpha)} \qquad (1 \leqslant i \leqslant p) \\ \qquad \nu_j(I_T^{(\alpha)}(\tilde{g}_j(X))) \geqslant \lambda_{p+j}^{(\alpha)} \geqslant \lambda_0^{(\alpha)} \qquad (1 \leqslant j \leqslant q) \\ \qquad X \in I_T^{(\alpha)}(\tilde{D}) \end{cases} \tag{4.2}$$

Where w_k are the weight (relative importance) among thecorresponding objective function and $\sum_{k=1}^{p+q} w_k = 1$.

Model (4.2) is essentially trying to use the weighted average operatorto allow for possible compensation among objectives and guarantees thatthe overall satisfactory degree of the compromise of the objectives isat least ta $\lambda_0^{(\alpha)}$. Also, model (4.2) should yield a α–Pareto solution.

In order to illustrate the proposed approach, we consider the following numeric example (omitted).

5. Conclusion

In this study, we have discussed a linear multiple objective programming problem with trapezoidal fuzzy numbers in the objectives and constraint coefficients. An auxiliary MOLP model is derived based on the ranking fuzzy number with total integral value. Furthermore, an extended version of Zimmermann's fuzzy approach has also been proposed to solve the auxiliary model for getting the α–Pareto solution. One of the advantage of the proposed approach is that the decision maker's degree of optimism has been taken into consideration.

Using the method of ranking fuzzy numbers with total integral values and the two phases fuzzy approaches proposed in this paper, we slaocan solve the multiple objective programming problem with imprecise objective and constraint coefficients as well as with the fuzzy equlities and fuzzy inequalities in the constraints.

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