Antisymmetrical intuitionistic fuzzy relation. Order on the referential set induced by an intuitionistic fuzzy relation

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Abstract: *In this paper we present an order on the referential set induced by an intuitionistic fuzzy relation of order. We will see how this induced order justifies the definition given of intuitionistic fuzzy antisymmetry.*

Keywords: *Intuitionistic fuzzy relation; composition of intuitionistic fuzzy relation; fuzzy relation; intuitionistic ordering relation; antisymmetrical intuitionistic property.*

1. Introduction

We start by remembering the definition given by K. Atanassov of intuitionistic fuzzy sets ([1]). Next, we give the definition of composition of intuitionistic fuzzy relations which was studied in ([5]). We finish this section remembering the most important properties of intuitionistic fuzzy relations on a set.

The second part of the paper is dedicated to presenting an order induced in the referential set X, through an intuitionistic fuzzy relation of order. The importance of this item is that it justifies the definition given of intuitionistic antisymmetrical relation, this definition does not have correspondence with the definition of antisymmetry for fuzzy case ([6]).

2. Preliminaries

Let X, Y and Z be ordinary finite non-empty sets.

Let $X \neq \phi$ be a given set. ([1]) An *intuitionistic fuzzy set* in X is an expression A given by

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\} \quad \text{where}$$
$$\mu_A : X \longrightarrow [0,1]$$
$$\nu_A : X \longrightarrow [0,1]$$

with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, for all $x \in X$

The numbers $\mu_A(x)$ and $\nu_A(x)$ denote respectively the degree of membership and the degree of non-membership of the elememt x in the set A. $IFSs(X)$ will denote the set of all the intuitionistic fuzzy sets in X. Obviously, when $\nu_A(x) = 1 - \mu_A(x)$ for every x in X, set A is a fuzzy set. We will call $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ *intuitionistic index* of the element x in the set A.

The following expressions are defined in ([2], [4], [5]) for every A, B belonging to $IFSs(X)$

1. $A \leq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$
2. $A \preceq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x)$ for all $x \in X$
3. $A = B$ if and only if $A \leq B$ and $B \leq A$
4. $A_c = \{< x, \nu_A(x), \mu_A(x) > | x \in X\}$

We know that an *intuitionistic fuzzy relation* is an intuitionistic fuzzy subset of $X \times Y$, that is, is an expression R given by

$$R = \{< (x,y), \mu_R(x,y), \nu_R(x,y) | x \in X, y \in Y\} \quad \text{where}$$

$\mu_R : X \times Y \longrightarrow [0,1]$ and $\nu_R : X \times Y \longrightarrow [0,1]$ satisfy the condition $0 \leq \mu_R(x,y) + \nu_R(x,y) \leq 1$ for every $(x,y) \in X \times Y$. $IFR(X \times Y)$ will denote the set of all the intuitionistic fuzzy subsets in $X \times Y$. The most important properties of intuitionistic fuzzy relations are studied in ([3], [5]).

Definition 1. *Let us consider $R \in IFR(X \times Y)$ and $P \in IFR(Y \times Z)$. We will call composed relation $P \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R \in IFRS(X \times Z)$ the one defined by*

$$P \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R = \{< (x,z), \mu_{P \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R}(x,z), \nu_{P \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R}(x,z) > | x \in X, z \in Z\} \quad \textit{where}$$

$$\mu_{P \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R}(x,z) = \bigvee_y \{\wedge[\mu_R(x,y), \mu_P(y,z)]\},$$

$$\nu_{P \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R}(x,z) = \bigwedge_y \{\vee[\nu_R(x,y), \nu_P(y,z)]\}$$

whenever $0 \leq \mu_{P \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R}(x,z) + \nu_{P \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R}(x,z) \leq 1,$ *for all* $(x,z) \in X \times Z$.

Now, we will remember the main properties of intuitionistic fuzzy relations in a set, that is, in $X \times X$. A complete study of these relations is made in ([5]).

Definition 2. *We will say that $R \in IFR(X \times X)$ is:*

1) Reflexive, if for every $x \in X$, $\mu_R(x,x) = 1$. Just notice that for every x in X, $\nu_R(x,x) = 0$.

2) Antisymmetrical intuitionistic, if

$$\text{for all } (x,y) \in X \times X, \quad x \neq y \ \text{ then } \begin{cases} \mu_R(x,y) \neq \mu_R(y,x) \\ \nu_R(x,y) \neq \nu_R(y,x) \\ \pi_R(x,y) = \pi_R(y,x) \end{cases}$$

3) Transitive, if $R \geq R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R$.

The definition given of intuitionistic fuzzy antisymmetry will be justified in this paper, as we will see later on.

Definition 3. *An intuitionistic fuzzy relation* R *on the cartesian set* $(X \times X)$, *is called an* ***intuitionistic order*** *if it is reflexive, transitive and antisymmetrical intuitionistic.*

3. Order on X induced by an IFR

Intuitionistic fuzzy relations can induce different relations in the universal set X. Now we are gong to study one of them.

Definition 4. *Let* R *be an element of* $IFR(X \times X)$, *we define a relation* $\preceq_R$ *in* X *through*

$$x \preceq_R y \ \text{ if and only if } \begin{cases} \mu_R(y,x) \leq \mu_R(x,y) \\ \nu_R(y,x) \geq \nu_R(x,y) \end{cases}$$

with $x, y \in X$.

Theorem 2. *If* $R \in IFR(X \times X)$ *is of intuitionistic order, then* $\preceq_R$ *is of ordinary order in* X.

Proof. i) $\preceq_R$ is reflexive because $\begin{cases} \mu_R(x,x) \leq \mu_R(x,x) \\ \nu_R(x,x) \geq \nu_R(x,x) \end{cases}$

ii) $\preceq_R$ is antisymmetrical because if $\begin{cases} x \preceq_R y \\ y \preceq_R x \end{cases}$ then $\begin{cases} \mu_R(y,x) = \mu_R(x,y) \\ \nu_R(y,x) = \nu_R(x,y) \end{cases}$ therefore $x = y$

iii) $\preceq_R$ is transitive, so that if $\begin{cases} x \preceq_R y \ \text{ with } x \neq y \\ y \preceq_R z \ \text{ with } y \neq z, \end{cases}$ we get

$$\begin{cases} \mu_R(y,x) \leq \mu_R(x,y) \\ \nu_R(y,x) \geq \nu_R(x,y) \end{cases} \qquad \begin{cases} \mu_R(z,y) \leq \mu_R(y,z) \\ \nu_R(z,y) \geq \nu_R(y,z) \end{cases}$$

firstly, let's see that they cannot occur at the same time

$$\begin{cases} \mu_R(z,x) \geq \mu_R(x,y) \\ \qquad \text{and} \\ \mu_R(z,x) \geq \mu_R(y,z) \end{cases}$$

In order to see it we suppose that if they occur at the same time

$$\mu_R(x,y) = \mu_R(z,x) \wedge \mu_R(x,y) \leq$$
$$\leq \bigvee_t [\mu_R(z,t) \wedge \mu_R(t,y)] = \mu_R(z,y).$$

$$\mu_R(y,z) = \mu_R(y,z) \wedge \mu_R(z,x) \leq$$
$$\leq \bigvee_t [\mu_R(y,t) \wedge \mu_R(t,x)] = \mu_R(y,x),$$

so

$$\mu_R(y,x) \leq \mu_R(x,y) \leq \mu_R(z,y) \leq \mu_R(y,z) \leq \mu_R(y,x)$$

$$\mu_R(z,y) \leq \mu_R(y,z) \leq \mu_R(y,x) \leq \mu_R(x,y) \leq \mu_R(z,y),$$

therefore

$$\mu_R(y,x) = \mu_R(x,y) = \mu_R(z,y) = \mu_R(y,z) = \mu_R(y,x)$$

$$\mu_R(z,y) = \mu_R(y,z) = \mu_R(y,x) = \mu_R(x,y) = \mu_R(z,y),$$

that is to say $\mu_R(x,y) = \mu_R(y,x) = \mu_R(z,y) = \mu_R(y,z)$ and as R is antisymmetric intuitionistic, we get $x = y$ and $y = z$ in opposition to the hypothesis, from where it is deduced that only one of the following possibilities can occur:

i) $\mu_R(z,x) < \mu_R(x,y)$ or ii) $\mu_R(z,x) < \mu_R(y,z)$

from i) it is deduced that

$$\mu_R(z,x) = \mu_R(z,x) \wedge \mu_R(x,y) \leq$$
$$\leq \bigvee_t [\mu_R(z,t) \wedge \mu_R(t,y)] = \mu_R(z,y) \leq \mu_R(y,z),$$

so

$$\mu_R(z,x) \leq \mu_R(x,y) \wedge \mu_R(y,z) \leq$$
$$\leq \bigvee_t [\mu_R(x,t) \wedge \mu_R(t,z)] \leq \mu_R(x,z)$$

from ii) we get $\mu_R(z,x) = \mu_R(z,x) \wedge \mu_R(y,z) \leq \bigvee_t [\mu_R(y,t) \wedge \mu_R(t,x)] = \mu_R(y,x) \leq \mu_R(x,y)$, therefore $\mu_R(z,x) \leq \mu_R(x,y) \wedge \mu_R(y,z) \leq \mu_R(x,z)$.

If we take a reasoning analogous to the previous one, we get for the non-membership functions that:

$$\begin{cases} \nu_R(z,x) \leq \nu_R(x,y) \\ \quad\text{and} \\ \nu_R(z,x) \leq \nu_R(y,z) \end{cases}$$

they cannot occur at the same time, so it happens that

i) $\nu_R(z,x) > \nu_R(x,y)$ or ii) $\nu_R(z,x) > \nu_R(y,z)$

from i) we deduce that $\nu_R(z,x) = \nu_R(z,x) \vee \nu_R(x,y) \geq \bigwedge_t [\nu_R(z,t) \vee \nu_R(t,y)] = \nu_R(z,y) \geq \nu_R(y,z)$,

therefore $\nu_R(z,x) \geq \nu_R(x,y) \vee \nu_R(y,z) \geq \nu_R(x,z)$.

From ii), reasoning in an analogous way, it is deduced that $\nu_R(z,x) \geq \nu_R(x,z)$. □

4. Remarks

Notice that our definition of intuitionistic antisymmetry is fundamental for the proof of this theorem and it does not recover the one by A. Kaufmann ([6]) for the fuzzy case. However, the advantage of the definition established by us is that, with this definition, if the intuitionistic relation is an intuitionistic fuzzy order, we can induce the order $\preceq_R$ in X, as we have proved in the previous Theorem. This fact does not happen with A, Kaufmann's definition.

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