DIFFERENCE AND SYMMETRIC DIFFERENCE OPERATORS BASED ON THE THEORY OF FALLING SHADOWS

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Abstract: By the use of the theory of falling shadows, we define difference and symmetric difference operators for fuzzy sets and obtain three kind of operators respectively.

Keywords: Fuzzy sets, random sets, falling shadows, difference, symmetric difference.

1. Introduction

In classical sets, symmetric difference of sets A and B is defined as

$$A\Delta B=(A\cap B^c)\cup(B\cap A^c) \quad (1)$$

As generalization of (1), there can be infinitely many possible ways to define the symmetric difference of fuzzy sets. In fact, let function $d:[0,1]^2\longrightarrow[0,1]$ satisfy:

$$d(0,0)=0,\ d(1,0)=1,\ d(0,1)=1,\ d(1,1)=0 \quad (2)$$

then for any fuzzy set A,B,

$$D(A,B)(u)=d(A(u),B(u)),\ \text{for any } u\in U \quad (3)$$

is a generalization of (1).

However, function d satisfying (2) is not only. Figure 1 can show that any surface to pass through the four point E,F,G,H can define a function d, and consequently can define a symmetric difference operators of fuzzy sets. Clearly, there are infinitely many such surfaces.

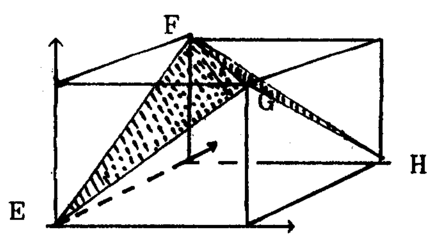

Figure 1

In this way, the various definition of symmetric defference operators for fuzzy sets can cause some perplexity. So a mathematical theory to support these definition should be developed.

In this paper, we shall establish a theoretical approach to define symmetric difference operators based on the theory of falling shadows presented by Wang[2], and obtain three new kind of operators(see theorem 2).

2. Theory of falling shadow

Theory of falling shadow, fuzzy sets being seen as falling shadows of random sets, is introduced by Goodman[1], Wang and Scanchez independently. Here, we shall give a brief description of the theory of falling shadows.

It was during 1981, under the supervision of Wang, that N.L.Zhang conducted a statistical experiment in China to determine the membership function of the fuzzy concept 'young' by three distinct groups of students. From that experiment, the resulting membership functions of 'young' were almost identical. This result signifies that the stability of the membership function of fuzzy concepts does exist in the theory of fuzzy sets. Wang then developed further his approach and presented a rigorous mathematical structure of his theory in his book entitled Fuzzy Sets and Falling Shadows of Random Sets[5]. The following is an extract from his book.

Given a universe of discourse U, for any $u \in U$, let

$$\dot{u}=\{A \mid u\in A \text{ and } A\subseteq U\}, \quad \dot{A}=\{\dot{u} \mid u\in A\} \qquad (4)$$

An ordered pair $(\mathbb{P}(U),\mathbb{B})$ is said to be hyper-measurable structure on U if $\mathbb{B}$ is σ-field in $\mathbb{P}(U)$ and $\dot{U}\subseteq\mathbb{B}$.

Given a probability space $(\Omega,\mathbb{A},P)$ and an hyper-measurable structure $(\mathbb{P}(U),\mathbb{B})$ on U, a random set on U is defined to be a mapping $\xi:\Omega\longrightarrow\mathbb{P}(U)$ that is $\mathbb{A}$-$\mathbb{B}$ measurable, that is

$$\forall C\in\mathbb{B},\ \xi^{-1}(C)=\{\omega \mid \omega\in\Omega \text{ and } \xi(\omega)\in C\}\in\mathbb{A} \qquad (5)$$

Suppose that ξ is a random set on U. Then the covering function of ξ, denoted by $\hat{\xi}$, is defined to be the probability of ω for which $u\in\xi(\omega)$, that is, $\hat{\xi}:U\longrightarrow[0,\ 1]$ where

$$\xi(u)=P(\omega \mid u\in\xi(\omega)), \text{ for each } u\in U \qquad (6)$$

ξ represents a fuzzy set A in U and we write $\hat{\xi}=A$. We shall call the random set ξ a cloud on A, and A the falling shadows of the random set ξ.

Once the probability space has been determined, then to each fuzzy set A in U, there corresponds a family of random sets whose falling shadows are all equal to A. Thus it is an important issue on how to choose a representative cloud for A. Indeed, the simplest one is the mapping $\xi:[0,\ 1]\longrightarrow\mathbb{P}(U)$ defined by

$$\xi:\lambda\longrightarrow A_\lambda\ ,\quad \lambda\in[0,1] \tag{7}$$

where A_λ is the λ-cut of A, that is $A_\lambda=\{u\in U|A(u)\geq\lambda\}$. It is not too difficult to show that ξ is a cloud of A. We shall call ξ defined above as the cut-cloud of A. This notion was firstly introduced by Goodman in [1].

3. Defining difference and symmetric difference of two fuzzy sets by falling shadows

Let A and B be fuzzy sets in the universe U. By [4], let $\mathbb{B}$ is a Borel field on [0, 1] and m the Lebesgue measure, the joint probability space is $([0,\ 1]^2,\mathbb{B}^2,P)$ and both of the projection of P on the unit interval [0, 1] are the Lebesgue measure m. Let

$$\xi:[0,\ 1]^2\longrightarrow\mathbb{P}(U)\qquad \eta:[0,\ 1]^2\longrightarrow\mathbb{P}(U)$$
$$(\lambda,\mu)\longrightarrow A_\lambda\qquad (\lambda,\mu)\longrightarrow B_\mu \tag{8}$$

Then we can obtain the following relation from the usual notion:

$$A_\lambda-B_\mu=A_\lambda\cap(B_\mu)^c\qquad A_\lambda\Delta B_\mu=(A_\lambda\cap(B_\mu)^c)\cup(B_\mu\cap(A_\lambda)^c) \tag{9}$$

$A_\lambda-B_\mu$, $A_\lambda\Delta B_\mu$ can be considered as random sets on U. By finding the shadow of two random sets, we can obtain the following definition:

Definition 1. For any two fuzzy sets A and B on U, difference of A and B is defined to be

$$(A-B)(u)=P((\lambda,\mu)|u\in A_\lambda-B_\mu) \tag{10}$$

Symmetric difference of A and B is defined to be

$$(A\Delta B)(u)=P((\lambda,\mu)|u\in A_\lambda\Delta B_\mu) \tag{11}$$

Theorem 1. Let $E_1=[0,\ A(u)]\times(B(u),\ 1]$, $E_2=[A(u),\ 1]\times(0,\ B(u)]$, then

$$(A-B)(u)=P(E_1),\qquad (A\Delta B)(u)=P(E_1\cup E_2) \tag{12}$$

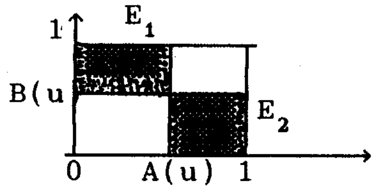

Figure 2

4. Marginal-uniform joint distribution on $[0, 1]^2$[4]

In theorem 1, P is a joint probability on $[0, 1]^2$, so different probability distribution P will generate different formulas. In general, if the probability space is

$$([0, 1]^2, \mathbb{B}^2, P) = ([0, 1], \mathbb{B}, m) \times ([0, 1], \mathbb{B}, m)$$

where $P(A\times[0, 1]) = m(A)$, $P([0, 1]\times A) = m(A)$, for any $A\in\mathbb{B}$, then there are infinitely many possibilities of joint distribution of P on the unit square $[0, 1]^2$. We shall refer P to as a marginal-uniform joint distribution on $[0, 1]^2$ and consider the following three basic cases.

I. *Perfect positive correlation*

If the whole probability P of (λ, μ) on $[0, 1]^2$ is concentrated and uniformly distributed on the main diagonal $I=\{(\lambda,\lambda)|\lambda\in[0, 1]\}$ of the unit square $[0, 1]^2$, then we say that the variables λ and μ are in perfect positive correlation.

II. *Perfect negative correlation*

If the whole probability P of (λ, μ) on $[0, 1]^2$ is concentrated and uniformly distributed on the anti-diagonal $\hat{I}=\{(\lambda,1-\lambda)|\lambda\in[0, 1]\}$ of the unit square $[0, 1]^2$, then we say that the variables λ and μ are in perfect negative correlation.

III. *Independent*

If the whole probability P of (λ, μ) on $[0, 1]^2$ is uniformly distributed on the unit square $[0, 1]^2$, then we say that the variables λ and μ are independent.

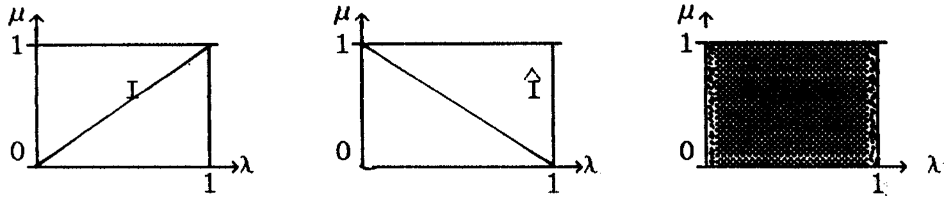

Fig.3a Fig.3b Fig.3C

5. Formulas of difference and symmetric difference

Now, we shall obtain three formulas of difference and symmetric difference under the above three basic correlations between the fuzzy sets A and B.

Theorem 2. (1) If the fuzzy sets A and B are in perfect positive correlation, then formula (12) will become

$$(A-B)(u)=A(u)-\min\{A(u),B(u)\}$$
$$(A\Delta B)(u)=\max\{A(u),B(u)\}-\min\{A(u),B(u)\} \quad (13)$$

(2) If the fuzzy sets A and B are in perfect negative correlation, then formula (12) will become

$$(A-B)(u)=\min\{A(u),1-B(u)\}$$
$$(A\Delta B)(u)=\min\{2-A(u)-B(u),\ A(u)+B(u)\} \quad (14)$$

(3) If the fuzzy sets A and B are independent, then formula (12) will become

$$(A-B)(u)=A(u)-A(u)B(u)$$
$$(A\Delta B)(u)=(1-B(u))A(u)+B(u)(1-A(u)) \quad (15)$$

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