**Integrals of Set-valued Functions for $\perp$-Decomposable Measure**

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**Abstract**

In this paper, basing on Weber's integrals for Archimedean t-conorms $\perp$-decomposable measures [5], integrals of set-valued functions for Archimedean t-conorms $\perp$-decomposable measures are established. Some results similar to Aumann's set-valued integrals are given. These include convexity, closedness, convergence theorems, etc. They are the generalization of Aumann's.

Keywords: measure theory, set-valued functions, t-conorm, $\perp$-decomposable measure, integral

## 1. Introduction

It is well known that set-valued functions have been used repeatedly in Economics [3], integrals of set-valued functions have been studied by Aumann[1], Debreu[2], and others. But they are all based on classical Lebesgue integrals. Weber [5] had introduced a kind of new integral for Archimedean t-conorms, (in short, $\perp$-integral) which is an extension of Lebesgue integrals and show much useful in fuzzy sets and statistics [6].

The paper's purpose is to extend the integrand of $\perp$-integrals from point-valued functions to set-valued functions, and build up a theory of $\perp$-integrals of set-valued functions, s.t. the $\perp$-integral of a point-valued function becomes special. Since $\perp$-integral is an extension of Lebesgue integrals, the paper's results are the generalization of Aumann's.

In the paper, following concepts and notations will be used. $(X, \mathscr{A})$ will denote a measurable space, $\perp$ is always a Archimedean t-conorm, g is the additive generator, m is a $\perp$-decomposable measure. $\int_A f \perp m$ is the resulting integral. the triplet $(X, \mathscr{A}, m)$ will still be a complete $\perp$-measure space. I denotes [0, 1]. $P_0(I)$ is the power set of I, which not including $\{\varphi\}$. A set-valued function is a mapping $F: X \to P_0(I)$. F is said to be measurable iff $GrF = \{(x, r): r \in F(x)\}$ is belong to $\mathscr{A} \otimes Borel(I)$. In the rest parts, the concepts undefined are all accepted from

Weber's [5].

## 2. Definitons and properties of $\perp$-integrals of set-valued functions.

In this section, $\perp$-integrals of set-valued functions will be defined by the similar way to Aumann's, then its properties are shown. These related to "convexity, closedess, montonicty".

2. 1. Definition. Let F be a set-valued function. The integral of F over $A \in \mathscr{A}$ is defined as

$$\int_A F \perp m = \{\int_A F \perp m : f \in S(F)\}$$

Where S (F) is the family of m-a. e. measurable selections of F. F is said to be integrable on A if $\int_A F \perp m \neq \varphi$.

2. 2. Lemma. For $\{R_n\} \subset P_0(I)$, if we define $\overset{\infty}{\underset{k=1}{\perp}} R_k = g^{(-1)}(\sum_{k=1}^{\infty} R_k) = g^{(-1)}$ $\{\sum_{k=1}^{\infty} r_k : r_k \in R_k,\ k \geqslant 1\}$, then $\int_A F \perp m$ can be divided into two cases:

(A): Except for (NSP) with m (A) = 1, then

$$\int F \perp m = g^{-1}(\int_A F\, d(g \cdot m))$$

(P) For (NSP) with m (A) = 1, X is assumed to be m-achievable, then

$$\int_A F \perp m = g^{(-1)}(\sum_{k=1}^{\infty} \int_{A_k \cap A} F d(g \cdot m) = \overset{\infty}{\underset{k=1}{\perp}} \int_{A_k \cap A} F \perp m$$

Where $\int_A Fd(g \cdot m)$ is the integral of Aumann's sense.

2. 3. Proposition. If F is a measurable set-valued function, then F is integrable on A.

Without any loss of generality, let the integral be over X, and "$\int_X$" is insteal by "$\int$".

A set-valued function F is said to be closed-valued (resp., convex-valued), if F (x) is closed (resp., convex) for $x \in X$ m-a. e. We use coF to denote the convex hull function of F. Next propositons is about convexity.

2. 4. Proposition. Let F be a measurable set-valued function. Then

(i) m is atomless $\Rightarrow \int F \perp m$ is convex;

(ii) F is convex-valued $\Rightarrow \int F \perp m$ is convex.

2. 5. Proposition. Let F be a measurable set-valued function. Then

$\int coF \perp m = co\int F \perp m$.

The above two propositions are related to convexity, then the next result is about closedness.

2. 6. Proposition. Let F be a measurable set-valued function with the assumption for case (S): there exists $\varphi$, s. t. $F(x) \leqslant \varphi(x)$ m-a. e., $\int \varphi \perp m < 1$ or case (NSP). $\sum_{k=1}^{\infty} (g \cdot m)(A_k) < \infty$. Then

F is closed-valued $\Rightarrow \int F \perp m$ is closed.

2. 7. Corollary. Let F be a measurable interval-valued function with the assumption in proposition 3. 7 i. e. $F(x) = [f^-(x), f^+(x)]$. Then

(i) $f^-, f^+ \in S(F)$

(ii) $\int F \perp m = [\int f^- \perp m, \int f^+ m]$

2. 8. Defintion. Let $R_1$ and $R_2$ be two subset of I. Then $R_1 \leqslant R_2$ iff

(i) For each $x_o \in R_1$, there exists $y_o \in R_2$, s. t. $x_o \leqslant y_o$;

(ii) For each $y_o \in R_2$, there exists $y_o \in R_1$, s. t. $x_o \leqslant y_o$.

Obviously, x, $y \in [0, 1]$, $x \leqslant y$ is a special case of this definition.

2. 9. Proposition. Let $F_1$ and $F_2$ be two measurable set-valued functions

If $F_1 \leqslant F_2$ (i. e. $F_1(x) \leqslant F_2(x)$ for $x \in X$ m-a. e.), then $\int F_1 \perp m \leqslant \int F_2 \perp m$.

## 3. Convergence theorems.

In this section, Fatou's lemmas, Lebesgue dominated convergence theorem, monotone convergence theorem are shown for the sequence of $\perp$-integrals of set-valued functions, we begin with the concept of convergence of a sequenee of elements in P (I).

3. 1. Definition. Let $\{R_n\}$ be a sequence of subsets of I. Define

$\text{Limsup } R_n = \{x : x = \lim_{k\to\infty} x_{n_k},\ x_n \in R_n (n \geqslant 1)\}$

$\text{Liminf } R_n = \{x : x = \lim x_n,\ x_n \in R_n (n \geqslant 1)\}$

If Limsup $R_n$ = Liminf $R_n$ = R, we say that $\{R_n\}$ converges to R, simply written by Lim $R_n$ = R or $R_n \to R$

3. 2. Note. Let $\{F_n\}$ be a sequence of set-valued functions. By the usual pointwise (m-a. e) way, we can define Limsup $F_n$, Liminf $F_n$ and Lim $F_n$.

3. 3. Theorem. (Fatou's lemmas) Let $\{F_n\}$ be a sequence of measurable set-valued function with the assumpsion for case (S): there exists $\varphi$, s. t. $F_n \leqslant \varphi (n \geqslant 1)$, $\int \varphi \perp m < 1$, or case (NSP): $\sum_{k=1}^{\infty} (g \cdot m)(A_k) < \infty$, resp. Then

(i) $\text{Limsup}\int F_n \perp m \subset \int \text{Limsup } F_n \perp m$;

(ii) $\int \text{Liminf } F_n \perp m \subset \text{Liminf}\int F_n \perp m$.

3. 4. Theorem. Lebesgue convergence theorem) Let the same conditions in theorem 3. 3 be given. If $F_n \to F$, then

$$\int Fn \perp m \to \int F \perp m$$

3. 5. Theorem. (Monotone convergence theorem) Let the same conditions in theorom 3. 3 be given. If $Fn \uparrow F$, then

$$\int Fn \perp m \uparrow \int F \perp m$$

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