WORKSHOP: ON MATHEMATICAL FOUNDATIONS OF FUZZY SET THEORY

General Foundational Remarks on Fuzzy Set Theory

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In the papers [1, 3, 4] we have initiated a nonstandard approach to fuzzy sets. In this workshop I want to summarise and make some additional remarks concerning the mathematical foundations of Fuzzy Sets.

1. **Mathematical Foundations.** When we say "Mathematical Foundations of Fuzzy Sets" we mean that, fuzzy set theory, should not be build up from scratch and using some aprioristic methods, but rather, we should start with existing foundations for classical mathematics, and then try to construct from then a non classical theory that contain, fuzziness and vagueness as a basic element. That is fuzziness should be build up from non - fuzzy classical mathematics, the same way that non - Euclidean Geometries are based on Euclidean. Presently there are the following options:

   (i) Base the transition on a non - classical deformation of Cantorian set theory, e.g. ZFC, to add up with a non- Cantorian set theory, which includes vagueness, fuzziness, etc. and is expressed using many - valued Logic. Options like Boolean - valued, Heyting - valued, MV - algebra valued models are possible pathways. In general the generalisation of the classical truth - value object, the trivial Boolean algebra $2 := \{0, 1\}$ and the study of various general truth - value objects are basic to the foundations of fuzzy sets.

   (ii) Understand the relationship of hypersets or non - well founded sets with fuzzy sets and with non - standard mathematics.

(iii) Give a categorical foundation of fuzzy sets, possibly in a bottom - up way, starting up with Infinitesimal models, passing to Boolean & Heyting valued models and topos theory and continuing to construct a qualitative analogue of these models, leading, perhaps, to MV - algebra - valued models end monoidal closed categories [6, 7].

2. **Representation and Interpretation.** It is well known that one can represent in a classical two - valued model of set theory, like e.g. the non - Neumann universe V, objects such as functions, probabilities, random sets, Boolean - valued sets, Fuzzy sets, etc. In this way, we can represent in V, in a crisp two - valued way, objects, which subsequently *we interpret* as multiple - valued objects. This interpretation is completely naive and very intuitive and consequently does not use the rich syntactic and semantic structures that exist at the formal level. The problem here is to formalise this interpretation in order to take advantage of these rich structures. So if we want to study formally fuzzy sets we should construct first a fuzzy universe of discourse, based on the representation of these objects in V. Similarly, for stochastic objects we should construct, stochastic universes, etc. For example, if $\mathbb{B}$ is a complete Boolean algebra then $V^{\mathbb{B}}$ is defined recursively as follows: $V_0^{\mathbb{B}} = \emptyset$ $V_{a+1}^{(\mathbb{B})} = \{v | dom(n) \subseteq V_a^{(\mathbb{B})} \& ran(n) \subseteq \mathbb{B}\}$ $V_\lambda^{(\mathbb{B})} = \bigcup_{a<\lambda} V_a^{(}\mathbb{B})$ if λ is a limit ordinal, and $V^{(\mathbb{B})} = \bigcup_{a+On} V_a^{(\mathbb{B})}$ If we substitute in the above definition, the Boolean truth value object $\mathbb{B}$ with objects like, [0,1] or A an MV-algebra, then we get other fuzzy universes.

3. **Quantitative vs. Qualitative.** It is imperative to make the distinction of quantitative vs. qualitative. We may say that Boolean-valued models express the qualitative aspects of randomness, whereas taking the probabilities of the truth values of statements, we have a quantitative expression of randomness. In respect to this we quote Halmos [5, p.186]: "... the mathematical theory of probability consists of the study of Boolean σ-algebras of sets.

   This is not to say that all Boolean σ-algebras of sets are within the domain of Probability theory. In general statements concerning such algebras and the relations between their elements are merely quantitative; probability theory differs from the general theory in that it studies also the quantitative aspects of Boolean algebras" In [2], we indicate that in a Boolean Power of a structure if we take probabilities and impose a logical structure on [0,1], by using the Lukasiewicz logical operators, then we convert a qualitative Boolean - valued structure into a quantitative MV - valued structure. A basic problem is : Can we represent an MV - algebra as a Boolean or Heyting algebra followed by an MV - algebra valued measure?

Related to this problem is the following:

Can we regard Boolean - valued, & Heyting - valued models and in general topos theory as the qualitative theory of fuzzy sets, whereas, MV-algebra valued models and in general monoidal closed categories as the quantitative part of the theory of Fuzzy sets? See [6, 7]

References

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