WORKSHOP: FOUNDATIONS OF FUZZY SET THEORY

FUZZY STRUCTURES AND COLLECTIONS OF LEVEL STRUCTURES

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One of the basic tools in classical mathematics, originating from human rules of thinking, is a two-element boolean algebra B_2. This structure is partially, even totaly ordered, it is a complete, distributive, complemented lattice, it could also be taken as a set of two real numbers with ordinary ordering. As a basis of a characteristic function of a subset, B_2 can thereby be generalized in different ways. The starting point is usually the ordering relation, as an interpretation of the set inclusion among characteristic functions.

Hence, depending on the codomain of the function from a set to the structure generalizing B_2, there are different kinds of fuzzy sets: real interval valued ([0,1]-valued), lattice valued (L-valued), boolean valued (B-fuzzy set), partially ordered fuzzy set (function from a set to a poset). There is also a relational valued fuzzy set, codomain of which is a relational system with a particular binary relation, generalizing this time the ordering itself (see References).

The common property of all functions - fuzzy sets is the existence of the uniquely determined family of level subset (cuts), ordered by set inclusion. For [0,1]-valued fuzzy sets this poset is a chain, for L- and B- valued ones it is a lattice. Importance of that family lies in the fact that the function - fuzzy set $\bar{A} : A \to P$ (A is a nonempty set and P a poset) can be reconstructed from the poset of its cuts, as shown by the well known formula

$$\bar{A}(x) = \bigvee_{p \in P} p \cdot A_p(x),$$

and similarly in the case of relational valued fuzzy sets. Moreover, $\bar{A}$ is usually connected with some mathematical (in the following we shall confine our attention to algebraic) structure A. As it is known, all the level functions as mathematical structures belong to the same class to which A belongs.

It is also known that the particular family of subsets of a nonempty set A can be considered as a collection of level functions of a fuzzy subset of A. Relational (ordering) properties of the family determine the nature of that fuzzy set. The same is with the family of substructures of the given (algebraic) structure A: they uniquely, as level functions, determine a fuzzy substructure of A.

Hence, the poset (or the relational structure) of cuts of some fuzzy substructure bears the whole complexity contained in the original structure. It seems that in the transition from a fuzzy set to the structure of its cuts, in a sort of a "defuzzification", nothing is lost.

The following simple example will illustrate the situation. Let G be the Klein's 4-group, with elements e, a, b and c. The "diamond"-lattice $SubG$ of its subgroups is given in Fig.1. Now, all the L-valued fuzzy subgroups of

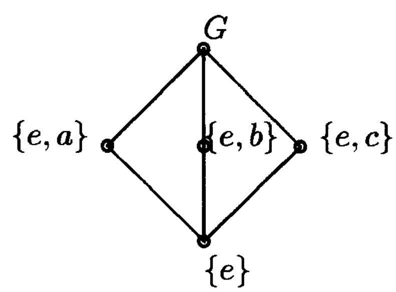

Fig.1

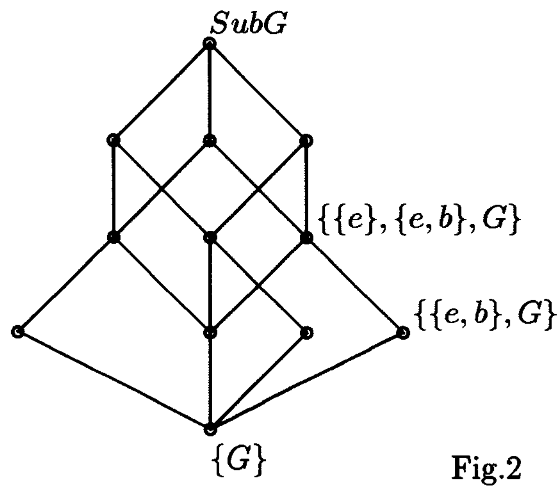

Fig.2

G (including the [0,1]-valued ones) are (up to the kind of an isomorphism, see References) determined by the Moore's families of subgroups from $SubG$. The poset (under $\subseteq$) of these families is a lattice represented in Fig.2. And all the P-valued fuzzy subgroups are (also uniquely) determined by th collections of subgroups such that for every x from G, the intersection of all subgroups in the collection containing x is also in that collection. The most generally, any collection of subgroups from G determines its relational valued fuzzy subgroup. And these are all fuzzy subgroups of G, in spite of the fact that there is an infinite number of posets P and mappings from G to P.

For example, all [0,1]-fuzzy subgroups of G can be described by the mappings

$$\bar{A} = \begin{pmatrix} e & a & b & c \\ 1 & p & q & q \end{pmatrix}, \bar{B} = \begin{pmatrix} e & a & b & c \\ 1 & q & p & q \end{pmatrix}, \bar{C} = \begin{pmatrix} e & a & b & c \\ 1 & q & q & p \end{pmatrix},$$

where p and q are two arbitrary real numbers from the interval [0,1], with $p \geq q$ (neutral element e, as a nullary function, has a value 1). These fuzzy subsets of G are uniquely determined by Moore's families from $SubG$, which are chains under set inclusion. The family $F = \{\{e\}, \{e,b\}, G\}$, for instance, is a three-element chain under the order reverse to $\subseteq$, and it determines the function $\bar{X} : G \to F$, such that for $x \in G$, $\bar{X}(x) = \bigcap(Y \in F \mid x \in Y)$.

Hence,

$$\bar{X} = \begin{pmatrix} e & a & b & c \\ \{e\} & G & \{e,b\} & G \end{pmatrix},$$

which is essentially the same mapping as $\bar{B}$.

The conclusion of the above discussion could be the following. To consider a fuzzy structure in all its complexity (fuzziness) means that one has to deal with properties of the collection of ordinary structures included in that complexity. It is also necessary to investigate relations (ordering and others) and all the operations among these structures. And there is always a question what is easier or more convinient to investigate: the mapping or the collection of its levels. The decision, of course, depends on the problem.

References

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