Further discussion on the Hahn decomposition theorem for signed fuzzy measure

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Abstract: This paper gives a counterexample that the union of two negative sets is not a negative set for signed fuzzy measure, showing that the proof of theorem 2 in [3] is not valid. Furthermore, we provide a proof for the Hahn decomposition theorem when the space X is countable or the signed fuzzy measure possess the property (d).

Keywords: Signed fuzzy measure; positive set; negative set; Hahn decomposition.

In[2],B.Jiao introducted a definition of signed fuzzy measure and proved a Hahn decomposition theorem for a finite signed fuzzy measure. But,some fuzzy measures in the sense of [4−8] are not signed fuzzy measures in the sense of [2] and a counterexample is given in [3]. Therefore, X.Liu introduced another definition of signed fuzzy measure as follows [3].

Definition 1. Let $(X, \mathcal{F})$ be a measurable space [1]. A set function $\mu : \mathcal{F} \to R \cup \{\pm \infty\}$ is called a signed fuzzy measure if μ satisfies

(1) $\mu(\emptyset)=0$;

(2) If $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$, $\{A_n\} \subset \mathscr{F}$, then $\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$;

(3) If $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$, $\{A_n\} \subset \mathscr{F}$, and $|\mu(A_n)| < +\infty$, $n \geqslant 1$, then $\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$;

(4) If $A, B \in \mathscr{F}$, $A \cap B = \emptyset$, then

(a) $\mu(A) \geqslant 0$, $\mu(B) \geqslant 0$, $\mu(A) \vee \mu(B) > 0 \implies \mu(A \cup B) \geqslant \mu(A) \vee \mu(B)$;

(b) $\mu(A) \leqslant 0$, $\mu(B) \leqslant 0$, $\mu(A) \wedge \mu(B) < 0 \implies \mu(A \cup B) \leqslant \mu(A) \wedge \mu(B)$;

(c) $\mu(A) > 0$, $\mu(B) < 0$, $\implies \mu(A) \geqslant \mu(A \cup B) \geqslant \mu(B)$.

In the following, we always suppose that μ is a signed fuzzy measure in the sense of Definition 1.

It follows easily from (a) and (b) that if $A_n \in \mathscr{F}$, $\mu(A_n) \geqslant 0$ (resp. $\mu(A_n) \leqslant 0$), $n \geqslant 1$ and there exists an n_0 such that $\mu(A_{n_0}) > 0$ (resp. $\mu(A_{n_0}) < 0$), then $\mu(\bigcup_{n=1}^{\infty} A_n) \geqslant \bigvee_{n=1}^{\infty} \mu(A_n) > 0$ (resp. $\mu(\bigcup_{n=1}^{\infty} A_n) \leqslant \bigwedge_{n=1}^{\infty} \mu(A_n) < 0$).

μ is said to be possessing property (d), if there exists a (finite or countable) sequence of sets $\{H_n\}$ in $\mathscr{F}$ such that the following conditions are satisfied

(i) $\forall n \geqslant 1, \mu(E)=0$ whenever $E \in \mathscr{F}$, $E \subset H_n$;

(ii) $\mu(\bigcup_{n=1}^{\infty} H_n) > 0$,

then $\mu(\bigcup_{t\in T} A_t) \geqslant 0$ whenever T is an arbitrary index set, $t \in T$, $A_t \in \mathscr{F}$ with (i) and $\bigcup_{t\in T} A_t \in \mathscr{F}$

Obviously, both any fuzzy measure [4–8] and any signed measure [1] are signed fuzzy measure possessing property (d).

The following proposition, which is similar to the property (d), play an important role in the proof of the Hahn decomposition theorem for signed fuzzy measure.

Proposition 1. If there exists a (finite or countable) sequence of sets $\{H_n\}$ in $\mathcal{F}$ such that the following conditions are satisfied

(i) $\forall_n \geq 1, \mu(E)=0$ whenever $E \in \mathcal{F}$, $E \subset H_n$;

(ii) $\mu(\bigcup_{n=1}^{\infty} H_n)>0$ (resp. <0),

then $\mu(\bigcup_{n=1}^{\infty} A_n) \geq 0$ (resp. ≤ 0) whenever a (finite or countable) sequence of sets $\{A_n\}$ in $\mathcal{F}$ with (i).

Proof. We only prove this proposition for "≥ 0", the rest may be proved in a similar way.

Suppose that $\{H_n\} \subset \mathcal{F}$ is a (finite or countable) sequence of sets with (i) and (ii). If there exists a (finite or countable) sequence of sets $\{A_n\}$ in $\mathcal{F}$ with (i) such that $\mu(\bigcup_{n=1}^{\infty} A_n)<0$, let $H_0=\bigcup_{n=1}^{\infty} A_n$, $E_n=H_n-\bigcup_{i=0}^{n-1} H_i$, $n \geq 1$. Evidently, H_0, E_1, $E_2 \cdots$ are pairwise disjoint and by (i), $\mu(E_n)=0$, $n \geq 1$. Hence, we have

$$\mu(\bigcup_{n=0}^{\infty} H_n)=\mu(H_0 \cup \bigcup_{n=1}^{\infty} E_n)<0 \qquad (*)$$

On the other hand, let $A_0=\bigcup_{n=1}^{\infty} H_n$, $F_n=A_n-\bigcup_{i=0}^{n-1} A_i$, $n \geq 1$. Obviously, A_0, F_1, $F_2, \cdots$ are pairwise disjoint and $\mu(F_n)=0$, $n \geq 1$. Noting that $\mu(A_0)>0$, We have

$$\mu(\bigcup_{n=0}^{\infty} A_n)=\mu(A_0 \cup \bigcup_{n=1}^{\infty} F_n)>0$$

which is a contradiction with (*) and so, for an arbitrary (finite or countable) sequence of sets $\{A_n\}$ in $\mathcal{F}$ with (i), $\mu(\bigcup_{n=1}^{\infty} A_n) \geq 0$.

The concepts of positive set and negative sets for signed fuzzy measure are exaxctly the same in form as the ones for classicial signed measure [1].

Definition 2. Let $E \subset X$. E is called a positive set (with respect to μ) if, for every $F \in \mathcal{F}$, $F \cap E \in \mathcal{F}$, $\mu(E \cap F) \geq 0$; Similarly E is called a negative set (with respect to μ) if, for every $F \in \mathcal{F}$, $F \cap E \in \mathcal{F}$, $\mu(E \cap F) \leqq 0$.

The empty set $\emptyset$ is both positive set and negative set in this sense, and every measurable subset of a positive set (resp. negative set) is also a positive set (resp. negative set).

Proposition 2. If A is a positive set (resp. negative set) and $\mu(A)=0$, then for every $E \in \mathcal{F}$, $E \subset A$, $\mu(E)=0$.

Proof. We only prove this proposition for positive set, the rest may be proved in a similar way.

Let A be a positive set and $\mu(A)=0$. If there exists a measurable set $E \subset A$ such that $\mu(E)>0$, since A is a positive set, $\mu(A-E) \geqq 0$. By Definition 1(a), we get

$$\mu(A)=\mu((A-E) \cup E) \geqq \mu(A-E) \vee \mu(E)>0$$

This is a contradiction with $\mu(A)=0$, and therefore, for every $E \in \mathcal{F}$, $E \subset A$, $\mu(E)=0$.

For every classical signed measure, if both A_1 and A_2 are negative sets, then so is $A_1 \cup A_2$. But, for a signed fuzzy measure, this conclusion is not true, as we can see in the following counterexample.

Counterexample. Let $X=\{a,b,c,d\}$, $\mathcal{F}$ be the power set of X, $\mu(\emptyset)=\mu(\{b\})=\mu(\{d\})=0$, $\mu(\{b,d\})=1$ and for the rest sets $A \in \mathcal{F}$, $\mu(A)=-1$. We can verify that μ is a signed fuzzy measure and both $\{a,b\}$ and $\{c,d\}$ are negative sets, But $\{a,b\} \cup \{c,d\}=X$ is not a negative set since $\{b,d\}$ in X is a positive set.

If let $\mu'(\emptyset)=\mu'(\{b\})=\mu'(\{d\})=0$, $\mu'(\{b, d\})=-1$ and for the rest sets $A \in \mathcal{F}$, $\mu'(A)=1$, then μ' is also a signed fuzzy measure and both $\{a,b\}$ and $\{c,d\}$ are positive sets, but $\{a,b\} \cup \{c,d\} =X$ is not a positive set since $\{b,d\}$ in X is a negative set. This show that the union of two positive sets is also not a positive set for signed fuzzy measure.

If there exist two sets A and B with $A \cap B = \emptyset$, $A \cup B = X$, such that A is a positive set and B is a negative set with respect to μ , then the sets A and B are said to be a Hahn decomposition of X (with respect to μ).

In [3], the Hahn decomposition of X with respect to a signed fuzzy measure is discussed and Theorem 2. is given. But, in the proof of Theorem 2. the wrong conclusion (that is, the union of negative sets is a negative set) is used, and therefore, the proof is mistaken. Naturally, one want to ask whether there still exists the Hahn decomposition of X with repect to a signed fuzzy measure ? In the following, we give some results on this problem.

Firstly, from Counterexample we find that there still exists the Hahn decomposition although the union of negative sets is not a negative set. Generally, we have the following theorem.

Theorem 1. Let X be a countable set, $\mathcal{F}$ be the power set of X and μ be a signed fuzzy measure on (X, $\mathcal{F}$),then there exists a Hahn decomposition of X with respect to μ.

Proof. Suppose $X=\{x_1, x_2, \cdots\}$. Let

$\mathcal{A}=\{x_i : \mu(\{x_i\})>0,\ x_i \in X\}$, $A=\cup\{x_i:\ x_i \in \mathcal{A}\}$;

$\mathcal{B}=\{x_i : \mu(\{x_i\})<0,\ x_i \in X\}$, $B=\cup\{x_i:\ x_i \in \mathcal{B}\}$;

$\mathcal{C}=\{x_i : \mu(\{x_i\})=0,\ x_i \in X\}$, $C=\cup\{x_i:\ x_i \in \mathcal{C}\}$.

If for arbitrary finite or countable $x_n \in \mathcal{C}$, $\mu(\cup_n x_n) < 0$, then it is easy to see that $B \cup C$ is a negative set, A is a positive set and $A \cap (B \cup C) = \emptyset$, $A \cup (B \cup C) = X$. And the conclusion follows.

If there exists finite or countable $x_n \in \mathscr{C}$, such that $\mu(\bigcup_n x_n)>0$, then by using Proposotion 1. $A \cup C$ is a positive set, B is a negative set and $(A \cup C) \cap B=\phi$, $(A \cup C) \cup B=X$. And the proof is completed

Secondly, in order to discuss the Hahn decomposition on general measurable space $(X, \mathscr{F})$ we need to add some weak conditions to signed fuzzy measure and need the following lemmas.

Lemma 1. Let μ be a signed fuzzy measure on $(X,\mathscr{F})$ and it satisfy

(1) $-\infty<\mu(A)\leqslant+\infty$, $\forall A \in \mathscr{F}$;

(2) $E \in \mathscr{F}$, $|\mu(E)|<+\infty \implies |\mu(F)|<+\infty$, $\forall$ $F \in \mathscr{F}$, $F \subset E$.

If $A \in \mathscr{F}$ and $\mu(A)<0$, then there exists a negative set $B \in \mathscr{F}$ such that $B \subset A$ and $\mu(B) \leqslant \mu(A)$.

Proof. See Lemma 1. in [3] or refer to the proof of Theorem 2.

Lemma 2. If there exists a (finite or countable) sequence of disjoint negative sets $\{N_n\}$ in $\mathscr{F}$, such that $X-\bigcup_n N_n$ is a positive set, then there exists a Hahn decomposition of X when one of the following conditions is satisfied.

(3) X is a countable set;

(4) μ is a signed fuzzy measure possessing property (d).

Proof. Suppose that $\{N_n\} \subset \mathscr{F}$ are finite or countable disjoint negative sets such that $X-\bigcup_n N_n$ is a positive set. Let $\mathscr{A}_n=\{H: H \in \mathscr{F}, H \subset N_n \text{ and } \mu(H)=0\}$, $K_n=\bigcup\{H: H \in \mathscr{A}_n\}$, $n>1$.

If for arbitrary $H_n \in \mathscr{A}_n$, $n>1$, $\mu(\bigcup_n H_n)<0$, let $B=\bigcup_n N_n$, $A=X-B$, then it is easy to see that B is a negative set, A is a positive set and the conclusion follows.

If there exists $H_n \in \mathscr{A}_n$, $n \geq 1$, such that $\mu(\cup_n H_n)>0$, let

$$B=\cup_n N_n-\cup_n K_n, \quad A=X-B=(X-\cup_n N_n)\cup \cup_n K_n$$

then B is a negative set and A is a positive set.

In fact, firstly for $E \in \mathscr{F}$, $EB \in \mathscr{F}$, by $\cup_n K_n=\cup_n N_k-B$, we get

$$E(\cup_n K_n)=E(\cup_n N_n)-EB \in \mathscr{F} ,$$

$$EB=E(\cup_n N_n)-E(\cup_n K_n)=\cup_n EN_n(E(\cup_n K_n))^c$$

(where A^c denotes the complement of A). Denote $F_n=EN_n(E(\cup_n K_n))^c$. It is clear that $F_n \subset N_n, F_n \in \mathscr{F}, n \geq 1$. Noting that N_n is a negative set, we have $\mu(F_n)\leq 0$, $n \geq 1$. If there exists some positive n. such that $\mu(F_{n.})<0$, then $\mu(EB)=\mu(\cup_n F_n)<0$; if $\mu(F_n)=0$, $n \geq 1$, noting the structure of K_n and B, then $F_n=\emptyset$, $n \geq 1$, and so $\mu(EB)=\mu(\cup_n F_n)$ $=\mu(\emptyset)=0$. Therefore, B is a negative set.

Next, for $E \in \mathscr{F}$ and $EA \in \mathscr{F}$, $EA=E(X-\cup_n N_n)\cup E(\cup_n K_n)$. Denote $F=E(X-\cup_n N_n)$, then $\mu(F) \geq 0$ since $X-\cup_n N_n$ is a positive set. By $EA \in \mathscr{F}$ and $F \in \mathscr{F}$, we get $E(\cup_n K_n)=\cup_n EK_n \in \mathscr{F}$, and further, $(\cup_n EK_n)\cap N_n=EK_n \in \mathscr{F}$, $n \geq 1$.

(i) Suppose that X is countable, then so is EK_n, $n \geq 1$. Thus there exists a (finite or countable) sequence of sets $\{H_{ni}\}$ in $\mathscr{A}_n$ such that $EK_n \subset \cup_i H_{ni}$, $n \geq 1$. It follows from Proposition 2. that

$$EK_n=\cup_i EK_n H_{ni}, \quad n \geq 1 \text{ and } \mu(EK_n H_{ni})=0, \ i \geq 1.$$

If $\mu(F)>0$, since $\mu(\cup_n H_n)>0$, by using Proposition 1. we have

$$\mu(E(\cup_n K_n))=\mu(\cup_n EK_n)=\mu(\cup_n \cup_i EK_n H_{ni}) \geq 0$$

and so $\mu(EA)=\mu(F\cup E(\cup_n K_n))>0$; if $\mu(F)=0$, then we get $\mu(EA)$ $=\mu(F\cup E(\cup_n K_n)) \geq 0$ from Propsition 1. immediately. Therefore, A is a positive set. This completes the proof of the lemma under the condition (3).

(ii) Suppose that μ is a signed fuzzy measure possessing property (d). Since for every $n \geqq 1$, EK_n is the union of certain sets in $\mathscr{A}_n$ and $\mu(\cup_n H_n)>0$, if $\mu(F)>0$, applying the property (d), we have $\mu(E(\cup_n K_n)) \geqq 0$ and $\mu(EA)=\mu(F \cup E(\cup_n K_n))>0$; if $\mu(F)=0$, we get $\mu(EA)=\mu(F \cup E(\cup_n K_n)) \geqq 0$ from the property (d) immediately. Therefore, A is a positive set. This completes the proof of the lemma under the condition (4).

Theorem 2. Let μ be a signed fuzzy measure on $(X, \mathscr{F})$ with the conditions (1) and (2) mentioned in Lemma 1., then there exists a Hahn decomposition of X when one of the conditions (3) and (4) mentioned in Lemma 2. is satisfied.

Proof. If for every $E \in \mathscr{F}$, $\mu(E) \geqq 0$, then μ is a fuzzy measure. We can take $A=X$, $B=\emptyset$, and the assertion follows.

If there exists $E \in \mathscr{F}$ such that $\mu(E)<0$, let $\delta_1=\inf\{\mu(E): E \in \mathscr{F}\}$ and choose $E_1 \in \mathscr{F}$ that satisfies $\mu(E_1)< \max(\frac{1}{2}\delta_1, -1)$, then $\delta_1<0$ and $\mu(E_1)<0$. By Lemma 1., there exists a negative set $N_1 \in \mathscr{F}$, $N_1 \subset E_1$, such that $\mu(N_1) < \mu(E_1)<0$.

If $X-N_1$ is a positive set, then the assertion follows from Lemma 2. Otherwise, there exists $E \in \mathscr{F}$, $E \subset X-N_1$, such that $\mu(E)<0$. Let $\delta_2=\inf\{\mu(E): E \in \mathscr{F}, E \subset X-N_1\}$ and choose $E_2 \in \mathscr{F}$, $E_2 \subset X-N_1$ that satisfies $\mu(E_2)<\max(\frac{1}{2}\delta_2, -1)$, then $\delta_2<0$ and $\mu(E_2)<0$. By Lemma 1., there exists a negative set $N_2 \in \mathscr{F}$, $N_2 \subset E_2$, such that $\mu(N_2) < \mu(E_2)<0$.

If $X-(N_1 \cup N_2)$ is a positive set, then the assertion follows from Lemma 2. Otherwise, we repeat the above processes.

Inductively, we have following two cases

(1) Either there exists some positive integer n, and a finite sequence of disjoint negative sets $N_1, \cdots, N_n$ in $\mathcal{F}$, such that $X - \cup_{i=1}^{n} N_i$ is a positive set.

(2) Or for arbitrary $n > 1$, $X - \cup_{i=1}^{n} N_i$ is not a positive set.

In the former case, the assertion follows from Lemma 2. immediately.

In the latter case, we obtain a sequence of real numbers $\{\delta_n\}$ and a sequence of disjoint negative sets $\{N_n\}$ in $\mathcal{F}$ that satisfy

$\delta_n = \inf\{\mu(E): E \in \mathcal{F}, E \subset X - \cup_{i=1}^{n-1} N_i\} < 0, \ n > 1$

$\mu(N_n) < \max(\frac{1}{2}\delta_n, -1) < 0, \ n > 1$ (**)

Supposing $X - \cup_{n=1}^{\infty} N_n$ is a positive set, then the assertion follows from Lemma 2.

Indeed, since $N_1, N_2, \cdots$ are pairwise disjoint, $\cup_{i=n}^{\infty} N_i \downarrow \phi$, and since

$$-\infty < \mu(\cup_{i=n}^{\infty} N_i) < \wedge_{i=n}^{\infty} \mu(N_i) < \mu(N_n) < 0, \quad n > 1$$

it follows from Definition 1 (3) that $\lim_{n\to\infty} \mu(\cup_{i=n}^{\infty} N_i) = 0$, and then $\lim_{n\to\infty} \mu(N_n) = 0$. Hence, by (**), we have $\lim_{n\to\infty} \delta_n = 0$. On the other hand, if $A \subset X - \cup_{n=1}^{\infty} N_n$, then for arbitrary $n > 1$, we have

$$A \subset X - \cup_{i=1}^{n-1} N_i, \quad \mu(A) > \delta_n \to 0.$$

and then $\mu(A) > 0$, that is, $X - \cup_{n=1}^{\infty} N_n$ is a positive set. This complete the proof of the theorem.

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