Some results on fuzzy prime spectrum of a ring

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ABSTRACT: In this note we extend the results in Kumar's paper "Fuzzy prime spectrum of a ring". In this regard we prove that the fuzzy prime spectrum of a ring and the elements of its basis are both compact. Then by giving two examples we will show that it is not true in general, that any element of a basis of fuzzy prime spectrum of a Boolean ring is closed, and the fuzzy prime spectrum itself is Hausdorff. Also by an example it is shown that the proof of the necessity of the condition of Theorem 5.3 of [3] is incorrect.

Keywords: Fuzzy (Prime) ideals, Fuzzy prime spectrum.

1. Preliminaries

In this section, some definitions, results, and notations which will be needed later on are presented. From now on R denotes a commutative ring with identity.

Definition 1.1. Let μ be a fuzzy subset of a set S and let $t \in [0,1]$. Then the set $\mu_t = \{x \in S | \mu(x) \geq t\}$ is called a level subset of μ. We let $\mu_* = \mu_1$, *i.e.* $\mu_* = \{x \in S | \mu(x) = 1\}$.

Note that if μ is a fuzzy ideal of R and $t \in [0, \mu(0)]$, then the level subset μ_t, $t \in Im(\mu)$ is an ideal of R and is called a level ideal of μ, where $Im(\mu)$

denotes the image of the function μ.

Theorem 1.2 [9,Proposition 2.1]. A fuzzy subset μ of R is a fuzzy ideal of R iff each level subset μ_t, $t \in Im(\mu)$ is an ideal of R.

Definition 1.3 [12,Definition 2.10]. Let A be a fuzzy subset of R. Then the fuzzy ideal generated by A, which is denoted by $< A >$, is defined by $< A >= \bigcap\{\mu | A \subseteq \mu, \mu$ is a fuzzy ideal of $R\}$.

Definition 1.4 [8,Definition 2.1]. A nonconstant fuzzy ideal μ of R is called fuzzy prime if for any two fuzzy ideal σ and θ of R the condition $\sigma\theta \subseteq \mu$ implies that either $\sigma \subseteq \mu$ or $\theta \subseteq \mu$.

Theorem 1.5 [8,Corollary 2.3]. A fuzzy ideal μ of R is fuzzy prime iff $Im(\mu) = \{1, t\}$, $t \in [0, 1)$ and the ideal μ_* is prime.

Definition 1.6. Let S be a set and $x \in S$. Then the fuzzy point x_β of S is a fuzzy subset of S which is defined by

$$x_\beta(y) = \begin{cases} \beta & \text{if } y = x \\ 0 & \text{otherwise }, \end{cases}$$

where $\beta \in (0, 1]$.

Corollary 1.7 [11,Lemma 3.4]. Let $x \in R, \lambda \in (0, 1]$. Then $(< x_\lambda >)^n =< x_\lambda^n >$, where $n \in \mathbb{N}$.

Theorem 1.8 [2]. If S is a multiplicative subset of R which is disjoint from an ideal I of R, then there exists a prime ideal ρ of R which is disjoint from S and containing I.

Notation 1.9. We let $X = \{\mu | \mu$ is a fuzzy prime ideal of $R\}$.

Definition 1.10. (See [3, Notation 2.7 (ii),(iii)]). Let θ be a fuzzy ideal of R. Then $V(\theta)$ and $X(\theta)$ are defined as follows:

$(i\ )\ V(\theta) = \{\mu \in X | \theta \subseteq \mu\}$,

$(ii)\ X(\theta) = X - V(\theta)$.

Definition 1.11 (See [3,Theorem 3.1]). Let $T = \{X(\theta)|\theta$ is a fuzzy ideal of $R\}$. Then the pair (X, T) is a topological space, and is called the fuzzy prime spectrum of R. It is denoted by F-spec R.

Lemma 1.12. (See the proof of Theorem 3.1 of [3]). Let $\{\theta_i\}_{i\in\Lambda}$ be a family of fuzzy ideals. Then

$$\bigcup_{i\in\Lambda} X(\theta_i) = X(<\bigcup_{i\in\Lambda} \theta_i >).$$

2. Main Results

Lemma 2.1. Let σ be a fuzzy subset of R. Then $V(<\sigma>) = V(\sigma)$. In particular $V(< x_\beta >) = V(x_\beta)$, for any fuzzy point x_β of R.

The following counter-example shows that the converse of Theorem 3.4(iii) of [3] is not true, in general.

Counter-example 2.2. Let $R = \mathcal{Z}$, $x = 1 \in \mathcal{Z}$, $\beta = \frac{1}{2} \in [0,1]$ and X=F-spec $\mathcal{Z}$. Define the $\mu \in X$ as follows:

$$\mu(x) = \begin{cases} 1 & \text{if } x \in 2\mathcal{Z} \\ 2/3 & \text{otherwise.} \end{cases}$$

Then $(x_\beta) \subseteq \mu$, hence $\mu \notin X(x_\beta)$, and consequently $X \neq X(x_\beta)$.

Lemma 2.3. Let $\beta_1, \beta_2 \in (0,1]$, $\beta = \min\{\beta_1, \beta_2\}$ and $x, y \in R$. Then

$$X(x_{\beta_1}) \bigcap X(y_{\beta_2}) = X((xy)_\beta).$$

Lemma 2.4. Let $K \subseteq (0,1]$, $\{x_i\}_{i\in\Lambda} \subseteq R$, and $X \subseteq \bigcup\{X((x_i)_t)|i \in \Lambda,\ t \in K\}$. Then $sup\{t|t \in K\} = 1$.

Lemma 2.5. Let $K \subseteq (0,1], \{x_i\}_{i\in\Lambda} \subseteq R,\ \alpha \in (0,1]$, and $x \in R$. If x is not nilpotent and $X(x_\alpha) \subseteq \bigcup\{X((x_i)_t) | i \in \Lambda, t \in K\}$, then $sup\{t | t \in K\} \geq \alpha$.

Lemma 2.6 Let $\alpha, \beta \in (0,1]$. If $\alpha \leq \beta$, then

i) $X(x_\alpha) \subseteq X(x_\beta)$

ii) $\vee(x_\beta) \subseteq \vee(x_\alpha)$.

Lemma 2.7. Let $K \subseteq (0,1], \beta = sup\{t | t \in K\}$, and $\{x_i\}_{i\in\Lambda} \subseteq R$. Then $\bigcup\{X((x_i)_t) | i \in \Lambda,\ t \in K\} = \bigcup\{X((x_i)_\beta) | i \in \Lambda\}$.

Lemma 2.8. Let $x_i \in R,\ i = 1, \ldots, n$, and $\beta \in [0,1]$. Then $X((\sum_{i=1}^{n} x_i)_\beta) \subseteq \bigcup_{i=1}^{n} X((x_i)_\beta)$.

Lemma 2.9. Let $\alpha \in (0,1],\ n \in \mathbb{N}$. Then $X(x_\alpha) \subseteq X((x^n)_\alpha)$.

Recall that a topological space Y is compact iff every covering of Y by basic open sets is reducible to a finite subcovering of Y.

Theorem 2.10. The topological space X is compact.

Theorem 2.11. Let $\alpha \in (0,1]$ and $x \in R$. Then $X(x_\alpha)$ is compact.

The following Theorem is a generalization of Theorem 3.6 of [3], which has been proved only for Boolean rings.

Theorem 2.12. Suppose that for any $x \in R$, there exists a positive integer $n_x \geq 2$ such that $x^{n_x} = x$, and $\alpha \in [0,1),\ \beta \in (0,1]$. If

$$A = \{\mu \in X | I_m(\mu) = \{1, \alpha\}\},$$

then the following statements hold:

(i) If $\beta > \alpha$, then $X(x_\beta) \bigcap A$ is both open and closed in A.

(ii) For any $x, y \in R$ there exists $z \in R$ such that

$$X(x_\beta) \bigcup X(y_\beta) = X(z_\beta)$$

(iii) A is Hausdorff.

Remark 2.13. It is known that (Exercise 23 on page 14 of [1]) in Zariski topology, if R is a Boolean ring, then any element X_f of basis of Spec R is both open and closed in Spec R. Moreover Spec R is Hausdorff.

Now the following examples show that these facts do not hold in general for $F - Spec\ R$. In other words there exists some element of basis of $X = F - Spec\ R$ which is not closed, and it is even possible that X is not T_1.

Example 2.14. Let $R = \mathcal{Z}_2$. Then

$$X = F - Spec\ R = \{\mu^t : t \in [0, 1)\}.$$

where μ^t is defined by

$$\mu^t(x) = \begin{cases} 1 & \text{if } x = \overline{0} \\ t & \text{if } x = \overline{1}. \end{cases}$$

Now we show that if $x = \overline{1}$ and $\alpha = \frac{1}{2}$, then $X(1_\alpha)$ is not closed. Suppose $X(1_\alpha)$ is closed. Then there exists a subset K of [0,1] such thst $X(1_\alpha) = \bigcap_{\beta \in K} V(y_\beta)$, $y \in \mathcal{Z}_2$. If $y = \overline{1}$ and $\beta \in (0, 1]$, then it is not difficult to check that $X(\overline{1}_\alpha) \not\subseteq V(\overline{1}_\beta)$. And if $y = \overline{1}$, $\beta = 0$ or $y = \overline{0}$, $\beta \in [0, 1]$, then it is seen that $V(y_\beta) = X$. Thus $X(1_\alpha)$ must be equal to X, which is a contradiction. Therefore $X(1_\alpha)$ is not closed.

Remark 2.15. Consider the fuzzy spectrum X of the Example 3.14. Choose $\mu^{1/2}$, $\mu^{1/3} \in X$. Let W be an open set containing $\mu^{1/2}$. Then $W = \bigcup_{\alpha \in K} X(1_\alpha)$, for some $K \subseteq (0, 1]$. Thus there exists $\alpha \in K$ such that $\mu^{1/2} \in X(1_\alpha)$. So $\alpha > \frac{1}{2} > \frac{1}{3}$. Consequently $\mu^{1/3} \in X(1_\alpha) \subseteq W$. In other words any open neighbourhood of $\mu^{1/2}$ also contains $\mu^{1/3}$. Thus X is not T_1, and in particular is not T_2.

Remark 2.16 Kumar [3,Theorem 5.3] has proved that if X is disconnected then R has a nontrivial idempotent. In his proof he asserts that $\sigma \oplus \theta =< \sigma \bigcup \theta >$. However as the following example shows, this is not necessarily true. Hence the statement of the theorem remains an open problem.

Example 2.17. Let $R = \mathcal{Z}_2$. Define the fuzzy ideals σ, θ of R as follows:

$$\sigma(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ 0 & \text{if } x = 1 \end{cases}$$

$$\theta(x) = \begin{cases} \frac{1}{3} & \text{if } x = 0 \\ 0 & \text{if } x = 1 \end{cases}$$

Then it can be seen that $\sigma \oplus \theta = \theta$ and $< \sigma \bigcup \theta >= \sigma$, thus $\sigma \oplus \theta \neq < \sigma \bigcup \theta >$.

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