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STRUCTURE OF FUZZINESS IN ALMOST TRAPEZOIDAL FUZZY QUANTITIES

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Key words: Fuzzy set, fuzzy quantity, linear space, trapezoidal fuzzy number, additive equivalence of fuzzy quantities.

If we call "almost trapezoidal" the fuzzy quantities which differ from the trapezoidal ones in "fuzzy zero" in the sense of [4] only then their set possesses some interesting algebraical properties, mostly mentioned in [3] and [4]. Here, we very briefly remember their essence, and discuss some of their consequences. Namely, any almost trapezoidal fuzzy quantity can be decomposed into (generally) four components, each of which is of specific type. It is possible to interpret the specificity of those components as rather different forms of fuzziness, and in this way to analyze the structure of fuzziness of the original almost trapezoidal fuzzy quantity. As such fuzzy quantities cover a wide class of fuzzy quantities with continuous membership function existing in actual applied problems, this analysis can offer an interesting view of the fuzziness.

1 Fuzzy Quantity

Let us denote by R the set of all real numbers. Any fuzzy subset a of R with the membership function $\mu_a : R \to [0,1]$ such that

(1) $$\exists\, x_0 \in R, \quad \mu_a(x_0) = 1,$$

(2) $$\exists\, x_1,\, x_2 \in R, \quad x_1 < x_2, \quad \forall\, x \notin [x_1, x_2], \quad \mu_a(x) = 0,$$

is called *fuzzy quantity*, and $\mathbb{R}$ denotes the set of all fuzzy quantities.

If $a \in \mathbb{R}$ then $(-a) \in \mathbb{R}$ denotes the fuzzy quantity fulfilling

(3) $$\mu_{-a}(x) = \mu_a(-x) \quad \text{for all } x \in R.$$

If $y \in R$ then $\langle y \rangle \in \mathbb{R}$ denotes the degenerated fuzzy quantity

(4) $$\mu_{\langle y \rangle}(y) = 1, \quad \mu_{\langle y \rangle}(x) = 0 \quad \text{for } x \neq y.$$

Due to [1] and other works, for $a,\, b \in \mathbb{R}$ and $r \in R$ we denote the sum $a \oplus b \in \mathbb{R}$ by

(5) $$\mu_{a \oplus b}(x) = \sup_{y \in R} \left(\min(\mu_a(y),\, \mu_b(x - y))\right), \quad x \in R,$$

and the crisp product $r \cdot a \in \mathbb{R}$ by

(6) $$\begin{aligned} \mu_{r \cdot a}(x) &= \mu_a(x/r) \quad \text{for } r \neq 0 \\ &= \mu_{\langle 0 \rangle}(x) \quad \text{for } r = 0,\ x \in R. \end{aligned}$$

There exist qualified reasons discussed, e. g., in [4] for the introduction of the following two concepts.

We say that $s \in \mathbb{R}$ is *0-symmetric* iff

$$\mu_s(x) = \mu_s(-x) \quad \text{for all } x \in R. \tag{7}$$

The set of all 0-symmetric fuzzy quantities is denoted by $\mathbb{S}_0$.

If a, $b \in \mathbb{R}$ then we say that they are *additively equivalent* and denote $a \sim_\oplus b$ iff there exist s_1, $s_2 \in \mathbb{S}_0$ such that

$$a \oplus s_1 = b \oplus s_2. \tag{8}$$

2 Trapezoidal Fuzzy Quantity

Fuzzy quantity a is called *trapezoidal* (cf. [1], [3]) iff there exists a quadruple of real numbers (a_1, a_0, a_0', a_2) such that $a_1 \le a_0 \le a_0' \le a_2$ and

$$\begin{aligned} \mu_a(x) &= (x-a_1)/(a_0-a_1) && \text{for } x \in [a_1, a_0], \\ &= 1 && \text{for } x \in [a_0, a_0'], \\ &= (a_2-x)/(a_2-a_0') && \text{for } x \in [a_0', a_2], \\ &= 0 && \text{for } x \notin [a_1, a_2]. \end{aligned} \tag{9}$$

It is useful to accept the following conventions. If $x_1 = x_0$ then

$$\begin{aligned} \mu_a(x) &= 0 \quad \text{for } x < x_1, \\ &= 1 \quad \text{for } x = x_1, \end{aligned}$$

and if $x_2 = x_0'$ then

$$\begin{aligned} \mu_a(x) &= 0 \quad \text{for } x > x_1, \\ &= 1 \quad \text{for } x = x_2. \end{aligned}$$

Any trapezoidal fuzzy quantity a is fully determined by the quadruple (a_1, a_0, a_0', a_2), called the *definitoric* one for a.

If $a \in \mathbb{R}$ is trapezoidal and (a_1, a_0, a_0', a_2) is definitoric for a then we say that a is *triangular* iff $a_0 = a_0'$. The definitoric quadruple of any triangular fuzzy quantity turns into the *definitoric triple* (a_1, a_0, a_2).

The most important concept discussed in the following sections of this paper is the one of *almost trapezoidal fuzzy quantity.* We use this term for any $a \in \mathbb{R}$ which is equivalent to some trapezoidal fuzzy quantity. It means that any $a \in \mathbb{R}$ is almost trapezoidal iff there exist b, s_1, s_2 such that s_1, $s_2 \in \mathbb{S}_0$, b is trapezoidal, and

$$a \oplus s_1 = b \oplus s_2, \quad \text{i.e.} \quad a \sim_\oplus b. \tag{10}$$

The set of all almost trapezoidal fuzzy quantities is denoted by $\mathbb{R}^{\mathrm{T}}$.

If $a \in \mathbb{R}^{\mathrm{T}}$ then we denote (using the notation of (10)) by $e \in \mathbb{R}$ the sum $e = a \oplus s_1$. Evidently $e \in \mathbb{R}^{\mathrm{T}}$ as $e = b \oplus s_2$, and we call it the *pure representation* of a. It means that each almost trapezoidal fuzzy quantity $a \in \mathbb{R}^{\mathrm{T}}$ has its pure representation $e = a \oplus s_1$

which can be decomposed into the sum of a trapezoidal fuzzy quantity b and a 0-symmetric component s_2,

(11) $$e = b \oplus s_2.$$

It is easy to see that the set $\mathbb{R}^T$ is closed regarding the equivalence relation $\sim_\oplus$ and the operations of addition (5) and crisp product (6); i.e. (cf. [3])

(12) $$a \in \mathbb{R}, \quad b \in \mathbb{R}^T, \quad a \sim_\oplus b \Longrightarrow a \in \mathbb{R}^T,$$

(13) $$a, b \in \mathbb{R}^T \Longrightarrow a \oplus b \in \mathbb{R}^T,$$

(14) $$r \in R, \quad a \in \mathbb{R}^T \Longrightarrow r \cdot a \in \mathbb{R}^T.$$

Evidently also $\langle y \rangle \in \mathbb{R}^T$ for any $y \in R$, as it is a special form of triangular fuzzy quantity, and, consequently, $\mathbb{S}_0 \subset \mathbb{R}^T$ as $s \sim_\oplus \langle 0 \rangle$ for any $s \in \mathbb{S}_0$.

The following properties of fuzzy quantities from $\mathbb{R}^T$ can be especially significant. For any a, b, $c \in \mathbb{R}^T$, any $s \in \mathbb{S}_0$, r, r_1, $r_2 \in R$

(15) $$a \oplus b \sim_\oplus b \oplus a,$$

(16) $$a \oplus (b \oplus c) \sim_\oplus (a \oplus b) \oplus c,$$

(17) $$a \oplus s \sim_\oplus a,$$

(18) $$a \oplus (-a) \sim_\oplus \langle 0 \rangle \sim_\oplus s,$$

(19) $$r \cdot (a \oplus b) \sim_\oplus (r \cdot a) \oplus (r \cdot b),$$

(20) $$(r_1 + r_2) \cdot a \sim_\oplus (r_1 \cdot a) \oplus (r_2 \cdot a),$$

(21) $$r_1 \cdot (r_2 \cdot a) = (r_1 \cdot r_2) \cdot a,$$

(22) $$1 \cdot a = a.$$

These properties mean that $\mathbb{R}^T$ is a commutative additive group (due to (15), (16), (17), (18)) and a linear space (due to (12),...,(22)) with the operations of addition and crisp product (5) and (6), respectively, with the additive equivalence relation $\sim_\oplus$ used instead of the equality, and with 0-symmetric fuzzy quantities from $\mathbb{S}_0$ representing the fuzzy zero (cf. [4]).

3 Decomposition

Let us consider a trapezoidal fuzzy quantity b with the definitoric quadruple (b_1, b_0, b'_0, b_2). Then there exist a triangular fuzzy quantity c with the definitoric triple (c_1, c_0, c_2) and a 0-symmetric trapezoidal fuzzy quantity i with the definitoric quadruple (i_1, i_0, i'_0, i_2) such that

(23) $$b = c \oplus i$$

and, $i \in \mathbb{S}_0$,

(24) $$i_1 = i_0 = (b_0 - b'_0)/2, \qquad i_2 = i'_0 = (b'_0 - b_0)/2,$$

(25) $$c_0 = (b_0 - b'_0)/2, \qquad c_1 = c_0 - (b_0 - b_1), \qquad c_2 = c_0 + (b_2 - b'_0).$$

It is evident that i is a crisp closed interval $[i_1, i_2]$.

The triangular fuzzy quantity c in (23) can be also decomposed in the following way. We write

$$c = u \oplus t, \tag{26}$$

where u and t are triangular fuzzy quantities, u is unilateral with definitoric triple (u_1, u_0, u_2), and t is 0-symmetric with the definitoric triple $(t_1, 0, t_2)$ constructed as follows.

If $c_0 - c_1 \geq c_2 - c_0$ then

$$t_1 = c_0 - c_2, \qquad t_2 = c_2 - c_0, \tag{27}$$

$$u_1 = c_1 + c_2 - 2c_0, \quad u_2 = u_0 = c_0. \tag{28}$$

If, on the contrary, $c_0 - c_1 < c_2 - c_0$ then

$$t_1 = c_1 - c_0, \qquad t_2 = c_0 - c_1, \tag{29}$$

$$u_1 = u_0 = c_0, \qquad u_2 = c_1 + c_2 - 2c_0. \tag{30}$$

Summarizing the previous considerations we can see that

$$b = u \oplus t \oplus i \tag{31}$$

where i is a 0-symmetric crisp interval, t is a 0-symmetric triangular fuzzy quantity and u is a non-symmetric (in the matter of fact unilateral) fuzzy quantity, eventually degenerated into a crisp number $\langle c_0 \rangle$ if $c_0 - c_1 = c_2 - c_0$.

The decomposition of trapezoidal fuzzy quantities can be easily extended to pure representations of the almost trapezoidal ones. Due to (10) and (12) the pure representation e of any $a \in \mathbb{R}^{\mathrm{T}}$ can be expressed as a sum $e = b \oplus s$ of a trapezoidal b and 0-symmetric $s \in \mathbb{S}_0$, with $s = \langle 0 \rangle$ if a itself is trapezoidal. Combining (12) and (31) any pure representation of an almost trapezoidal fuzzy quantity $a \in \mathbb{R}^{\mathrm{T}}$ can be decomposed into

$$e = u \oplus t \oplus i \oplus s \tag{32}$$

with the specification of particular components given above. Each of those fuzzy quantities can be degenerated into a crisp numbers (t, i, s can degenerate into $\langle 0 \rangle$, u into $\langle (b_0 + b_0')/2 \rangle$).

4 Interpretation

The almost trapezoidal fuzzy quantities can be interesting because of the linearity of their set $\mathbb{R}^{\mathrm{T}}$ mentioned in Section 2 ((15),…,(21)), which is valid if the strict equality between fuzzy quantities is substituted by the additive equivalence $\sim_\oplus$ (in certain sense "equality up to fuzzy zero", c.f. [4]). The set $\mathbb{R}^{\mathrm{T}}$ is relatively large and the variety of its elements offers a possibility to approximate quite wide class of fuzzy quantities representing real data in various applications.

The fact of the possibility of decomposition spontaneously provokes the question if and in what measure it reflects some more fundamental structure of fuzziness hidden in the formal properties of almost trapezoidal fuzzy quantities. The author's opinion is that there really exists some structuralization of fuzziness, and that the decompositions (31) and (32) point at some of its (maybe quite subtile) features. Let us note that the fact of the decomposition of fuzziness, on rather more general but also rather more rough level, was briefly discussed in [5], already.

Let us consider the fuzzy quantities on the right-hand-side of (32).

First: the membership functions of some of them are composed from linear segments (it concerns u, t, i), the remaining one needs not be linear. This is a specific property of quantities from $\mathbb{R}^{\mathrm{T}}$ – they are defined to be such. Anyhow, it illustrates the fact that the formal presentation of some, theoretically quite interesting, fuzzy quantities can include linear and non-linear component.

Second: It is much more interesting to consider the input components of the sum (32) from the point of view of their 0-symmetry and uncertainty. In any $a \in \mathbb{R}^{\mathrm{T}}$ there exist the following types of fuzziness which can be derived from its pure representation e:

- An "interval fuzziness" (in fact fully deterministic and crisp), represented by i, which marks the basic extent of the modal values (values with possibility equal to 1); in certain sense the dispersion of the determinism included into the modelled data. It can be described as 0-symmetric, i. e. with balanced weight of values being larger or smaller than the mean.
- A "balanced variable fuzziness", represented by t and s. Both of them are 0-symmetric, i. e. their fuzziness is balanced, both types of values (larger and smaller than the mean) are distributed symmetrically. Both, t and s, are "fuzzy zeros" and in this sense they do not influence the linear algebraic operations regarding the $\sim_{\oplus}$ equivalence of inputs and outputs. Anyhow, each of them reflects another structure of uncertainty:
  - t – represents the linear, regularly increasing or decreasing possibilities of the relevant values,
  - s – generally breaks the linearity of this component, and includes certain type of fluctuations of uncertainty (e. g. more or less possible extremal values, discontinuity of possibilities, their convexity, etc.) described by nonlinearities in the membership function of s (and, consequently, of a and e).
- The "non-symmetric fuzzines", represented by u, showing if (and in what extent) the uncertainty "sides" the values either smaller or greater than the mean. Moreover, its modal value determines the real mean value of all uncertainties mentioned above.

It is worth mentioning that the properties of $s \in \mathbb{S}_0$ in (10) were not further specified. Generally, s itself may be a composition of some triangular or trapezoidal 0-symmetric part and some "more irregular" non-linear 0-symmetric component; in symbols $s = s' \oplus b'$, where s' can be considered for the representative of some more concentrated irregularity, and b' can be processed analogously to b in (11), having all its properties.

If any of the components of the right-hand-side of (32) degenerates into a crisp number, the resulting almost trapezoidal fuzzy quantity e looses some of its typical features. It can be unimodal (if i degenerates) and becomes trapezoidal, it looses non-linearities of membership function (if s degenerates), it looses asymmetry (if u degenerates), and its membership function becomes discontinuous (if μ_s is not continuous).

5 Uniqueness

The decomposition (31) of any trapezoidal fuzzy quantity b into the triple

$$b = u \oplus t \oplus i$$

is unique, and all its components are exactly defined by the formulas introduced in Section 3.

The decomposition of an almost trapezoidal fuzzy quantity e which is a pure representation of some other almost trapezoidal fuzzy quantities is not generally unique. First, the same quantity $e \in \mathbb{R}^{\mathrm{T}}$ can be pure representation of more almost trapezoidal fuzzy quantities,

$$e = a \oplus s_1 = a' \oplus s_1', \quad a \neq a', \; a, \, a' \in \mathbb{R}^{\mathrm{T}}.$$

Second, for any $a \in \mathbb{R}^{\mathrm{T}}$ there may exist two (or more) 0-symmetric fuzzy quantities $s_1, \, s_1' \in \mathbb{S}_0$ such that

$$a \oplus s_1 = a \oplus s_1' = e,$$

and analogously there exist different $s_2, \, s_2'$ or such that

$$e = b \oplus s_2 = b \oplus s_2'.$$

Example 1. Let b be trapezoidal with definitoric quadruple (b_1, b_0, b_0', b_2) being

$$b_1 = -2, \quad b_0 = -1, \quad b_0' = 1, \quad b_2 = 2.$$

Then for any $s \in \mathbb{S}_0$ such that

$$\begin{aligned} \mu_s(x) &= x + 2 && \text{for } x \in [-2, -1], \\ &= x - 2 && \text{for } x \in [1, 2], \\ &= \text{arbitrary} && \text{for } x \in (-1, 1), \\ &= 0 && \text{for } x \notin [-2, 2] \end{aligned}$$

the sum $b \oplus s$ is a trapezoidal (it means also almost trapezoidal, and being representative of (almost) trapezoidal b) fuzzy quantity $e = b \oplus s$ with definitoric quadruple (e_1, e_0, e_0', e_2)

$$e_1 = -4, \quad e_0 = -2, \quad e_0' = 2, \quad e_2 = 4.$$

It means that $e = b \oplus s$ for many $s \in \mathbb{S}_0$ of the described type and, analogously $e = a \oplus s$ for the same class of s if we put $a = b$ or $a = b \oplus \langle 0 \rangle$.

Example 2. Using the idea of the previous example, we may put $s \in \mathbb{S}_0$ trapezoidal, with the definitoric quadruple $(-2, -1, 1, 2)$ and define almost transversible $a, \, a' \in \mathbb{R}^{\mathrm{T}}$ in the following way

$$a = t \oplus s', \qquad a' = t \oplus s'',$$

where t is triangular with the definitoric triple $(-1, 0, 1)$ and

$$\begin{aligned}\mu_{s'}(x) &= -x && \text{for } x \in [-1, 0],\\ &= x && \text{for } x \in [0, 1],\\ &= 0 && \text{for } x \notin [-1, 1],\\ \mu_{s''}(x) &= 1 - \sqrt{1 - x^2} && \text{for } x \in [-1, 1],\\ &= 0 && \text{for } x \notin [-1, 1].\end{aligned}$$

Then $a \neq a'$ as, for example,

$$\mu_a(0) = \frac{1}{2}, \qquad \mu_{a'}(0) = 1 - \sqrt{2}/2.$$

But,

$$\begin{aligned}\mu_a(x) = \mu_{a'}(x) &= x + 2 && \text{for } x \in [-2, -1],\\ &= 2 - x && \text{for } x \in [1, 2],\\ &= 0 && \text{for } x \notin [-2, 2],\end{aligned}$$

and, as shown in Example 1, already, $a \oplus s = a' \oplus s$.

It is easy to see that the same pure representation e of some almost trapezoidal $a \in \mathbb{R}^{\mathrm{T}}$ can be decomposed into a trapezoidal and 0-symmetric component, in more ways, namely $e = b \oplus s_2 = b' \oplus s_2' = \ldots$, as well.

6 Reduction of Almost Trapezoidal Fuzzy Quantities

The character of the addition operation $\oplus$ implies that its repetitive application inevitably (with the exception of addition of the degenerated quantities (4)) increases the output fuzziness. It means that the extent of the support set of $a \oplus b$ is larger than those ones of a and b. In this way the range of possible values of some fuzzy quantities can be unpleasantly large, where most of the resulting uncertainty, or fuzziness, appears as a result of cumulation of fuzzy noise present on the input of some calculations.

In such case it could be quite frequently desirable to reduce the extent of fuzziness of some fuzzy quantities. Of course, the optimal form of such reduction depends on the demands of each particular application of the theory. The decomposition of trapezoidal and almost trapezoidal fuzzy quantities, given by (31) and (32), respectively, offers one of such reductions. Namely, if we accept the approach considering 0-symmetric fuzzy quantities for non-essential fuzzy zeros, it is evident that any of the component t, i, s in (31) and (32) can be omitted, and in this way the range of the decomposed fuzzy quantities (in formulas (31) and (32) denoted by b and e, respectively) can be even essentially reduced. It depends on the interpretation and on the modelled real situation which one of them (or even all) can be omitted without loss of essential information.

If b, b' are two trapezoidal fuzzy quantities decomposed due to (31) into

$$b = u \oplus t \oplus i, \qquad b' = u' \oplus t' + i'$$

then for their sum

$$b \oplus b' = (u \oplus u') \oplus (t \oplus t') \oplus (i \oplus i')$$

the components $(t \oplus t')$ and $(i \oplus i')$ keep their shape of 0-symmetric and triangular or crisp interval, respectively. It is not generally true regarding the sum $u \oplus u'$, which ned not be unilateral if

$$u_0 - u_1 > 0 \quad \text{and} \quad u'_2 - u'_0 > 0.$$

As in any case $u \oplus u'$ is triangular, its decomposition into unilateral and 0-symmetric components, analogous to the procedure described in (27), (28) and (29), (30), can be easily done. In such case the extent of fuzziness (the support set of the membership function) of $u \oplus u'$ is decreased again, and the procedure contributes to the limitation of the scale of significant possible values of $b \oplus b'$.

7 Conclusive Comments

The remarks presented above suggest rather a discussion topic than a compact theory. Nevertheless, two comments are worth mentioning explicitely.

First, there still exists the problem of construction of the membership functions of trapezoidal and almost trapezoidal fuzzy quantities. Inspirative in this sense can be the approach presented e. g. in [2] and based on the idea of fuzzy similarity relation generated by some monotonous function. It can be found interesting that even in this case the symmetry gains certain degree of significancy. It could be easy to derive the shape of the monotonous linear function (and corresponding fuzzy relation of similarity) generating triangular, trapezoidal and semi-trapezoidal fuzzy numbers.

The second comment presented here concerns the relation between almost trapezoidal fuzzy quantity and its pure representation. Let $a \in \mathbb{R}^{\mathrm{T}}$ be almost trapezoidal, it means that there exists trapezoidal b and 0-symmetric $s_1, s_2 \in \mathbb{S}_0$ such that

$$a \oplus s_1 = b \oplus s_2.$$

If $e = a \oplus s_1$ is the pure representation of a then there exists at least one additive decomposition of e into trapezoidal and 0-symmetric part, namely, $e = b \oplus s_2$. The open question is if for such a and e there exists another decomposition of e, let us denote it

$$e = b' \oplus s' \oplus s_1,$$

where $b' \oplus s' = a$. It means if any almost trapezoidal $a \in \mathbb{R}^{\mathrm{T}}$ itself can be expressed as a sum of trapezoidal and 0-symmetric component without using the auxiliary concept of pure representation.

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