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Reduction costs as a vital argument in favour of fuzzy models

1. Introduction

Fuzzy set literature usually justifies the application of fuzzy numbers by the fact, that they allow an adequate mathematical shaping of inaccuracy, which is not of stochastic nature. Instead of using "average numbers" for only vaguely known data; fuzzy numbers and fuzzy intervalls make it possible to model the subjective imaginations of a decision maker as precise as he can express them. When modelling real problems by means of fuzzy systems the chances for getting a wrong picture of reality and by that selecting a solution which does not correspond to the original problem will immensely be reduced. This circumstance signifies a deciding factor and illustrates one considerable advantage fuzzy models can offer.

However, another and in my opinion even more important advantage of fuzzy modells is frequently ignored in literature. Deterministic and stochastic models require a vast amount of information to ensure the identification of at least acceptable "average numbers", so that the probability of incorrect modelling is kept as low as possible. After the calculation process it regularly becomes evident, that the major part of information was not essentially necessary for determining the solution and therefore it would be sufficient to work with vague data in these sectors. This obvious conflict is caused by the fact, that in classical models the judgement when information must be precise and at which point vague data will be satisfactory can only be made after a solution has been found and that implies an ex post-decision.

This paper will demonstrate through the analysis of various decision problems, that fuzzy modelling in combination with an interactive solution process presents an adequate answer to the information dilemma of real problems. Instead of an extensive gathering of information ex ante, the aquisition of additional information will be oriented at set aims and carried out under consideration of cost-benefit-relations.

2. Modelling decision problems by means of standard decision support or classical optimisation systems

In case a real decision problem shall be designed through standard decision support or optimisation models, the decision maker has to look out for a suitable typ of model. Then the parameters of the chosen model have to be specified as accurate as possible, as with incorrectly fixed parameters the chances of solving a model which does not illustrate the real decision problem increase and that implies a calculated solution which represents no reasonable decision proposal.

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The straight transfer of a real problem into a standard model requires precisely circumscribed parameters, a qualification which is also indispensable for the distribution function of stochastic models. Those specifications can only be based on a tremendous amount of information, which has to be gathered ex ante, and is therefore linked with immense information costs. Books on operations research usually undervalue this fact, as they concentrate their special interest on solution algorithms. However the problem can not be ignored, that even with a huge effort und high information costs there is no guarantee that all parameters can be precisely defined. Particular difficulties occur with numbers which will realize in future. This circumstance presents a vital argument for justifying the application of fuzzy models and is therefore often mentioned in fuzzy literature.

Let us suppose we succeeded in designing a good model of the real problem and calculate a appropriate solution. We then take a closer look at those constraints or alternatives which finally determined the solution. For the majority of cases we will find that only the parameters of the constraints which causally influenced the solution or the values of favoured alternatives actually needed a clear-cut specification. Vague specifications would satisfy the requirements expected by the rest of the model parameters. The problem we have is that we can only identify the relevant facts **after** the calculation of the solution.

Although those particular relations can be recognized more often with an increasing complexity of the model, we will demonstrate the fundamental aspects by means of two simple examples:

< **1** > The simple linear maximisation problem

$$x_0 = 300x_1 + 400 + 100 \rightarrow \text{Max}$$

subject to

$$\begin{aligned} x_1 - x_2 &\leq 8 \\ 2x_1 + 3x_2 &\geq 44 \\ 2x_1 - 5x_2 &\geq -36 \\ 4x_1 + 5x_2 &\geq 80 \\ 4x_1 + 3x_2 &\leq 97 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Figure 1 illustrates that the optimal solution is found in (14.5, 13), a point which is confirmed by calculating the intersection point of the 3. and the 5. constraint boundary line. Above that it is obvious that the point (14.5, 13) would represent the optimal solution, even if the parameter of the 1., 2. and the 4. constraints had been specified by slightly altered numbers.

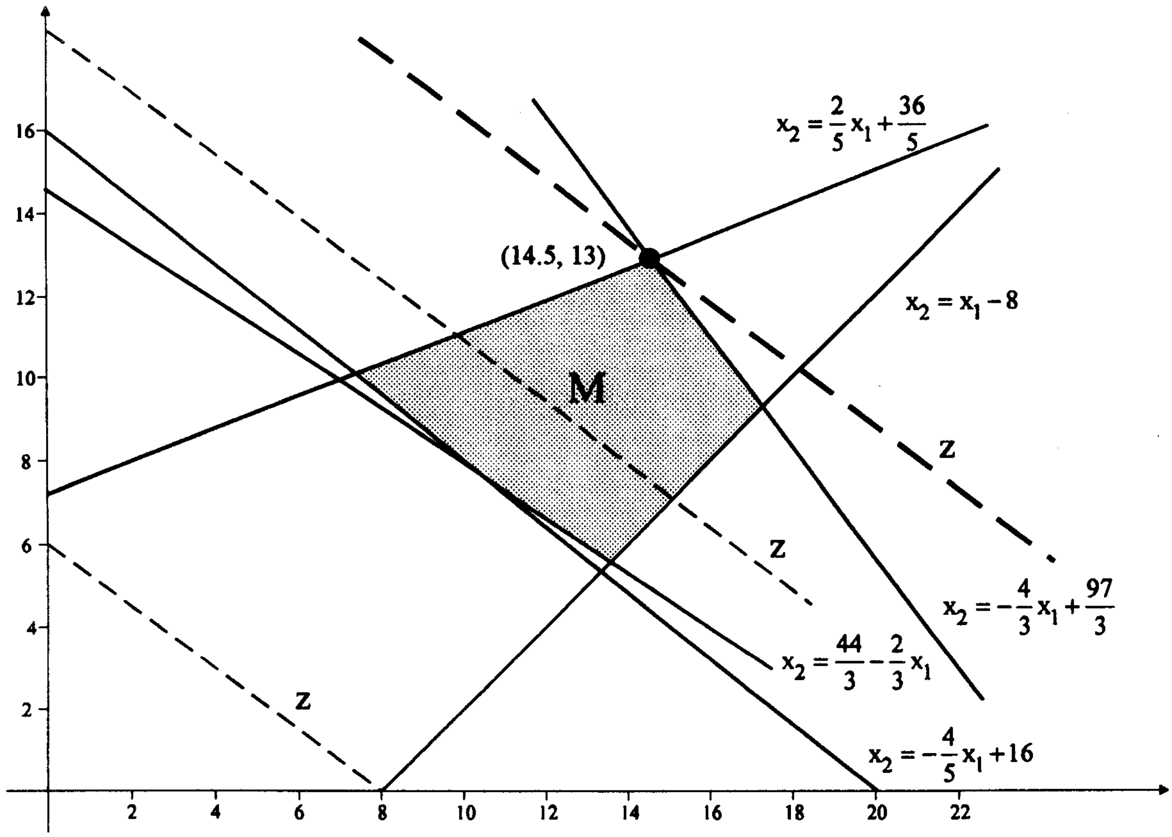

Fig. 1: Graphic solution of the linear programming problem

< **2** > A simple decision between alternatives under uncertainty (expected value criterion)
A producer has the problem to determine the output of a product. Based on his pattern of production he has the choice between five alternatives which are put in order according to size:

$a_1 < a_2 < a_3 < a_4 < a_5$.

The profit earned with a specific output depends on the demand, which is not with absolute certainty known. Due to his amount of information the producer either considers a „high" (state of nature s_1), an „average" (state of nature s_2) or a „low" (state of nature s_3) demand. He appoints the following a priori-probabilities to those states of nature:

$p(s_1) = 0.5,\ p(s_2) = 0.3,\ p(s_3) = 0.2$.

The suceeding profit matrix displays which profits measured in 1.000 DM correspond to the alternative constellations of output and demand.

	s_1	s_2	s_3	expected payoff
a_1	210	100	-80	119
a_2	170	110	-60	106
a_3	150	140	-10	115
a_4	110	100	0	85
a_5	50	50	50	50

table 1: A priori-payoff matrix

Now we have the problem of calculating the solution of the fuzzy LP-model. Today there exist several efficient solution programs for fuzzy linear optimisation models which can be employed for establishing a first solution, see LAI; HWANG (1992), ROMMELFANGER (1994A).

The solution of the described fuzzy maximisation problem shown in table 2 has been calculated by the recently developed solution algorithm FULPAL 2.0, see ROMMELFANGER (1994A), p. 262ff.

```
2 variables,   5 constraints,   1 objectives                 Lambda = 0.921505

VARIABLES

        x(1)     =  14.769901
        x(2)     =  12.461782

OBJECTIVES

1.               (     9415.6831     9415.6831     9415.6831     9415.6831 )
Rhs              (     9334.8511     9426.4706 )
Asp. level                   9357.7560                      μ(1) = 0.921505

CONSTRAINTS

1.               (       -0.6459        2.3081        2.3081        5.2621 )
Rhs                                          (        8.0000        9.5000 )
Asp. level                                                 9.1250
p = 0                                                       μ(1) = 1.000000

2.               (      -80.7718      -71.3561      -61.9404      -55.2479 )
Rhs                                          (      -44.0000      -40.0000 )
Asp. level                                               -41.0000
p = 0                                                       μ(2) = 1.000000

3.               (       26.5382       30.2768       32.7691       39.0000 ) -->>
Rhs                                          (       36.0000       39.0000 )
Asp. level                                                38.2500
p = 0                                                       μ(3) = 1.000000

4.               (     -133.5274     -122.8655     -113.4498     -101.3110 )
Rhs                                          (      -80.0000      -73.0000 )
Asp. level                                               -74.7500
p = 0                                                       μ(4) = 1.000000

5.               (       84.0953       91.0186       97.9419      104.8653 )
Rhs                                          (       97.0000      105.0000 )
Asp. level                                               103.0000
p = 0                                                       μ(5) = 0.921505
```

table 2: First solution of the fuzzy LP-system

The solution $(x_1, x_2) = (14.769901; 12.461782)$ results in the maximum value 9415.6831. The solution is determined by the constraint 3 (at 0-level) and the constraint 5 (at 1-level).

In order to obtain a more precise solution, one now has to look for additional information which presumably improves the specification of the constraints which actually determine the solution. Assuming that additional information indicates that the crisp coefficients and the exact right hand sides should be fixed like in the classical LP-model of example < 1 >. Then a new calculation of the more precisely described fuzzy LP-model presents the identical optimal solution as the classical LP-model, see table 3. It is irrelevant that the constraints 1, 2 and 3 are still only vaguely outlined.

```
2 variables,   5 constraints,   1 objectives                   Lambda = 1.000000


VARIABLES

         x(1)      =  14.500000
         x(2)      =  13.000000


OBJECTIVES

1.                (     9550.0000     9550.0000     9550.0000     9550.0000 )
Rhs               (     9550.0000     9550.0000 )
Asp. level                     9550.0000                     μ(1) = 1.000000


CONSTRAINTS

1.                (       -1.4000        1.5000        1.5000        4.4000 )
Rhs                                           (        8.0000        9.5000 )
Asp. level                                                   9.1250
p = 0                                                          μ(1) = 1.000000

2.                (      -81.9000      -72.3500      -62.8000      -56.0000 )
Rhs                                           (      -44.0000      -40.0000 )
Asp. level                                                 -41.0000
p = 0                                                          μ(2) = 1.000000

3.                (       36.0000       36.0000       36.0000       36.0000 ) -->>
Rhs                                           (       36.0000       36.0000 )
Asp. level                                                  36.0000
p = 0                                                          μ(3) = 1.000000

4.                (     -135.3000     -124.4500     -114.9000     -102.6000 )
Rhs                                           (      -80.0000      -73.0000 )
Asp. level                                                 -74.7500
p = 0                                                          μ(4) = 1.000000

5.                (       97.0000       97.0000       97.0000       97.0000 ) -->>
Rhs                                           (       97.0000       97.0000 )
Asp. level                                                  97.0000
p = 0                                                          μ(5) = 1.000000
```

table 3: Second solution of the fuzzy LP-system

Obviously it makes sense to determine the objective function right from the beginning as precise as possible. However, information costs also play an important part. In the case of high information costs we therefore recommend to start with calculating a compromise solution of the optimisation problem with vague coefficients in the objective function in order to find out which values the objective function can possibly present.

At that stage the decision maker has to decide whether the costs for additional information will be worth the effort. The solution algorithm FULPAL, but also other algorithms, for example FLIP of R. Slowinski (1990), offer an appropriate instrument for calculating not only compromise solutions of fuzzy LP-models with fuzzy constraints and an objective function with fuzzy coefficients, but FULPAL is also suitable for multiobjective models. As every real number can be interpreted as a specific fuzzy number the coefficients may be real or fuzzy.

Regarding the assumed a priori-probability distribution it is evident that the altenatives a_4 and a_5, eventually even alternative a_2 come off a lot worse than the alternatives a_1 and a_3. Even an improvement of their specific evaluations of the state of nature would not increase their chances for being chosen as first preference.

3. Fuzzy decision support and optimisation systems for modelling decision problems

One way to limit the extensive information process could be that one starts designing a model of the real problem with only the information which can be obtained with little effort and at reasonable costs. For some of the parameters of the model no exact and workable specification will be found. Those vague numbers could be replaced by „means" entailing however the problem that by that a model may be conceived which does not adequately picture the real problem. Therefore it is more advantageous to admit the vague data into the model and then try to calculate the solution.

A possibility to include vague or verbal evaluations into mathematical models offers the fuzzy set theory. That is why first of all the transformation of real decision problems into fuzzy modells is recommended.

3.1 Modelling through fuzzy optimisation models

To begin with we look at a LP-problem with a standard objective function and constraints which may contain trapezoid fuzzy intervals or triangular fuzzy numbers. These special (flat) fuzzy numbers will be abbreviated as $\tilde{A} = (\underline{a} - \underline{\alpha}; \underline{a}; \bar{a}, \bar{a} + \bar{\alpha})$ and $\tilde{B} = (b - \underline{\beta}; b; b + \bar{\beta})$, see figure 2.

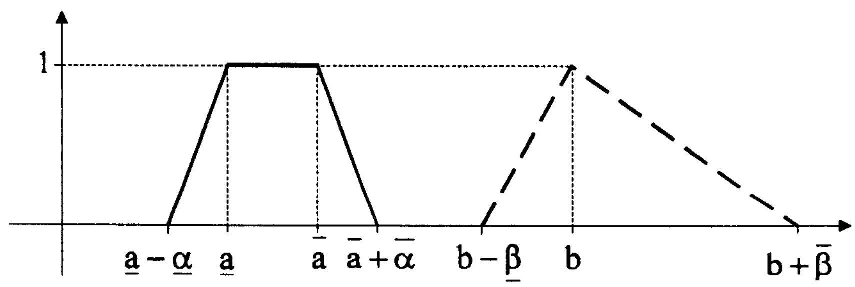

figure 2: Membership functions of $\tilde{A}$ and $\tilde{B}$

<3> $x_0 = 300\,x_1 + 400\,x_2 + 100 \rightarrow \text{Max}$

subject to

$$
\begin{array}{rrcr}
(0.8;\,1;\,1;\,1.2)\,x_1 & -\,x_2 & \leq & (8;\,8;\,9.5) \\
(1.8;\,2;\,2.3;\,26)\,x_1 & +\,(2.3;\,2.6;\,3;\,3.4)\,x_2 & \geq & (40;\,44;\,44) \\
2\,x_1 & -\,(4.5;\,4.8;\,5;\,5.2)\,x_2 & \geq & (-39;\,-36;\,-36) \\
(3.4;\,3.8;\,4.1;\,4.4)\,x_1 & +\,(4.1;\,4.6;\,5;\,5.5)\,x_2 & \geq & (73;\,80;\,80) \\
(3.5;\,3.8;\,4.1;\,4.4)\,x_1 & +\,(2.6;\,2.8;\,3;\,3.2)\,x_2 & \leq & (97;\,105;\,105) \\
 & x_1,\,x_2 & \geq & 0
\end{array}
$$

As a comparison of the objective values in table 2 and 3 reveals, the calculated solutions of the fuzzy LP-models do not give precise guidance about how additional information influences the objective values. However, they allow at least a rough estimation of the size of the objective values. This knowledge can then be applied to cost-benefit-considerations when decisions about gathering additional information have to be made.

3.2 Modelling through fuzzy decision models

< 2a >

	s_1	s_2	s_3	expected payoff
a_1	(170; 200; 220; 230)	(70; 90; 110; 120)	(-110; -90; -70; -50)	(84; 109; 119; 131)
a_2	(140; 160; 175; 190)	(85; 100; 115; 125)	(-85; -70; -50; -40)	(78.5; 96; 112; 124.5)
a_3	(120; 140; 160; 170)	(115; 135; 145; 150)	(-30; -20; 0; 10)	(87; 105; 123.5; 132)
a_4	(85; 105; 115; 125)	(85; 95; 105; 115)	(-15; -10; 10; 15)	(65; 79; 91; 100)
a_5	(45; 50; 55; 60)	(40; 45; 50; 55)	(35; 45; 50; 60)	(41.5; 47.5; 52.5; 56.5)

table 4: A priori-payoff matrix of a producer with trapezoid payoffs $\tilde{U}(a_i, s_j)$

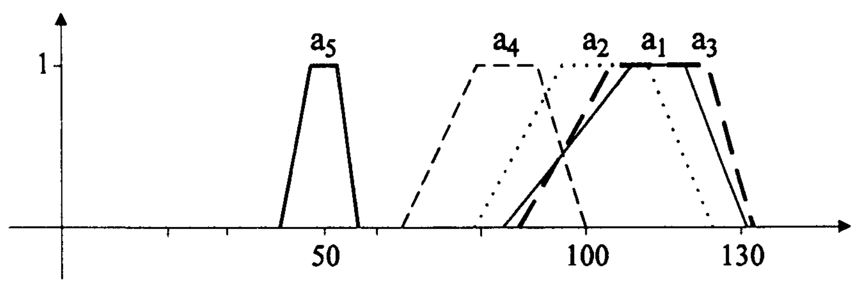

figure 3: Fuzzy expected payoffs

In table 4 and figure 3 it becomes evident, that as long as the postulated a priori-distribution is accepted the alternative a_5 and the alternative a_4 and a_2 will not be taken into consideration. Assuming the a priori-distribution is correct extra information about the preferred alternatives a_1 and a_3 regarding their specific evaluations of the state of natures should be gathered. Just to make sure, one could additionally attempt to improve the evaluations of a_2 and a_4; in case the former information process did not supply the wanted selectivity the taking up of information can also be carried out within a second step. However, an additional gathering of information only makes sense for the case that the information costs are not higher than
Max (87 - 84; 109 - 105; 123.5 - 119; 132 - 131) = 4.5.

We now presume that the extra information results in the following more precise evaluation of the alternatives a_1 and a_3 , see table 5 and figure 4.

	s_1	s_2	s_3	expected payoff
a_1	(205; 212; 220; 225)	(93; 98; 106; 113)	(-96; -87; -79; -65))	(111.2; 118; 126; 133)
a_3	(134; 145; 152; 160)	(126; 135; 140; 145)	(-19; -10; -3; 5)	(101; 111; 117.4; 124.5)

table 5: A priori-payoff matrix of a producer with trapezoid payoffs $\tilde{U}(a_i, s_j)$ *and additional information*

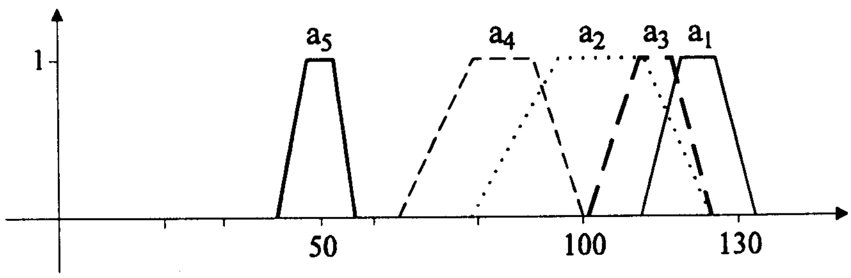

figure 4: Fuzzy expected payoffs including additional information

After having taken up information alternative a_1 can be identified as the optimal solution under the postulated probability distribution.

Another possibility to obtain a superior selectivity for the alternatives offers the classical method which examines a test market in order to substitute the a priori-probability distribution by a posteriori-probabilities, see ROMMELFANGER (1994A), p. 97ff.

Still one has to consider the case that extensive information about the entry of the states of nature may not be available. So it could occur that the a priori-probabilities are not described precisely, but only vaguely by means of fuzzy intervals, see ROMMELFANGER (1994A), p. 120ff. Then fuzzy expected values could be calculated, whereby it has to be understood that such a weak state of information results in relatively vague evaluations; therefore less alternatives can be recognised as being dominated and excluded from the examination.

4. Conclusions

The preceding examples illustrate that the modelling of real decision problems by means of fuzzy models leads to a reduction of information costs; that circumstance is caused by the fact that within the interactive solution process additional information is gathered by orientation at the requirements and under consideration of cost-benefit-relations. Therefore we recommend to start with transferring the real problem into a fuzzy model instead of trying to design an operable model right from the beginning. The fuzzy model will then be reduced to an operable model in order to calculate compromise solutions and by that to present additional information. That information will be regarded as the basis for further decisions about the taking up of additional information and reveals in which details the fuzzy modell has to be altered to find a better adjustment to the real problem.

Those observations can be transferred to various models; besides others fuzzy net plans can be named in this context, see ROMMELFANGER (1994B).

5. Selected Literature

LAI Y.J.; HWANG C.L. (1992) Fuzzy Mathematical Programming - Methods and Applications. Springer-Verlag Heidelberg Berlin

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