THE CONVEXITY AND PIECEWISE LINEARITY OF THE FUZZY CONCLUSION GENERATED BY LINEAR FUZZY RULE INTERPOLATION

László T. Kóczy
LIFE Chair of Fuzzy Theory
Dept. of System Science, Tokyo Institute of Technology
4259 Nagatsuta, Midori-ku, Yokohama 227, Japan
(On leave from the Dept. of Telecommunication and Telematics,
Technical University of Budapest, Sztoczek u. 2., Budapest H-1111, Hungary.)
and
Szilveszter Kovács
Computer Centre, University of Miskolc
Miskolc-Egyetemváros, Miskolc, H-3515, Hungary
e-mail: szkszilv@gold.uni-miskolc.hu

1. The method of linear rule interpolation

In fuzzy modeling by If...then rules a crucial problem is the large (exponential) size and the consequently high computational time of reasoning algorithms. One solution for decreasing the size of the model is the use of *sparse rule bases* where

$$\bigcup_{i=1}^{r} \mathrm{supp}(A_i) \subset X$$

Here $\boldsymbol{R} = \{R_1, \ldots, R_r\}$ the fuzzy rule base, $R_i = A_i \rightarrow B_i$ the ith fuzzy rule
and $X = X_1 \times X_2 \times \ldots \times X_n$ the input universe of discourse.

In the gaps of the sparse rule base, the use of the classical reasoning algorithms [1,2,3] is impossible, because the observation $\boldsymbol{x}$ does not intersect with any of the rule antecedents:

$$\boldsymbol{x} \cap \bigcup_{i=1}^{r} \mathrm{supp}(A_i) = \varnothing$$

The method of **linear rule interpolation** is one of the possibilities of approximate reasoning in the sparse rule bases. It decomposes the problem of fuzzy approximation into an infinite family of crisp problems, corresponding to the α-cuts of the rules and the observation. It solves the interpolation for every α independently and deduces the fuzzy solution by uniting these results into a fuzzy approximation again. For details of this method see e.g. [4,5,6,7], for an extension to multilevel rule bases see e.g. [8].

The basic form of fuzzy rule interpolation is the **linear interpolation of two fuzzy rules**, defined by:

$$\mathrm{dist}(\boldsymbol{A}_1,\boldsymbol{x}) : \mathrm{dist}(\boldsymbol{x},\boldsymbol{A}_2) = \mathrm{dist}(\boldsymbol{B}_1,\boldsymbol{y}) : \mathrm{dist}(\boldsymbol{y},\boldsymbol{B}_2)$$

where $\boldsymbol{A}_1 \prec \boldsymbol{x} \prec \boldsymbol{A}_2$ and $\boldsymbol{B}_1 \prec \boldsymbol{B}_2$

$R_i = \boldsymbol{A}_i \rightarrow \boldsymbol{B}_i \quad i \in [1,2]$ the fuzzy rules flank the observation $\boldsymbol{x}$

The dist($\boldsymbol{F}$,$\boldsymbol{G}$) denotes the fuzzy distance between the fuzzy sets $\boldsymbol{F}$ and $\boldsymbol{G}$. The complete information on the fuzzy distance is two extended "fuzzy sets", $d_L^{\alpha}(\boldsymbol{F},\boldsymbol{G})$ and $d_U^{\alpha}(\boldsymbol{F},\boldsymbol{G})$ which are two families of distances (corresponding to the α-cuts) between $\inf\{\boldsymbol{F}_\alpha\}$, $\inf\{\boldsymbol{G}_\alpha\}$ and $\sup\{\boldsymbol{F}_\alpha\}$, $\sup\{\boldsymbol{G}_\alpha\}$ (e.g.Fig.1.).

If the universe of discourse of the fuzzy sets $\boldsymbol{F}$,$\boldsymbol{G}$ is multidimensional, the distances between $\inf\{\boldsymbol{F}_\alpha\}$, $\inf\{\boldsymbol{G}_\alpha\}$ and $\sup\{\boldsymbol{F}_\alpha\}$, $\sup\{\boldsymbol{G}_\alpha\}$ can be defined in the Minkowski sense: $d_L^{\alpha}(\boldsymbol{F},\boldsymbol{G}) = \left(\Sigma^{k}_{i=1} d_L^{\alpha}(F_i, G_i)^{w} \right)^{1/w}$

There are certain necessary conditions for defining fuzzy distances between fuzzy sets. One is the existence of full ordering in every component of the universe of discourse of the fuzzy sets, and as a consequence, the existence of a partial ordering $\langle$ in the universe of discourse (graduality of the components). The other one is the existence of distances in every component of the universe of discourse of the fuzzy sets.
A further important restriction is that all the comparable fuzzy sets should be convex and normal, otherwise some α-cuts are not connected or do not exists at all, which makes the distance corresponding to these α-cuts meaningless.

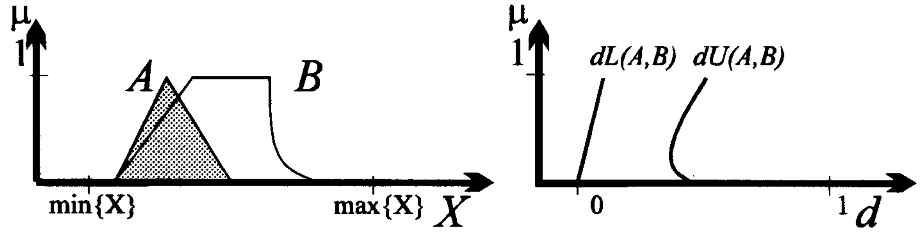


Fig. 1. Fuzzy distance between the fuzzy sets A and B, $d_L{}^{\alpha}(A,B)$ and $d_U{}^{\alpha}(A,B)$

The linear interpolation of two fuzzy rules deals only with two rules from the rule base. It is necessary for the interpolation that the two rules R_1 and R_2 are comparable both in their antecedents and consequents and that they flank the observation (also in the sense of $\langle\rangle$): $A_1 \langle x \langle A_2$ and $B_1 \langle B_2$
The partial ordering $\langle$ as a precedence < or > can have a different meaning on the side of antecedents and consequents. The only important thing is the flanking of the observation on the antecedent side and the existence of the ordering on the consequent side (the consequents of the rules must be comparable in the sense of the ordering $\langle\,\rangle$).
With the extended fuzzy sets of fuzzy distances the fundamental equation is written as:

$$d_L{}^{\alpha}(A_1,x):d_L{}^{\alpha}(x,A_2) = d_L{}^{\alpha}(B_1,y):d_L{}^{\alpha}(y,B_2)$$
$$d_U{}^{\alpha}(A_1,x):d_U{}^{\alpha}(x,A_2) = d_U{}^{\alpha}(B_1,y):d_U{}^{\alpha}(y,B_2) \qquad \forall\, \alpha \in [0,1]$$

where A_1,A_2,B_1,B_2,x convex and normal fuzzy sets, d_L the lower fuzzy distance and d_U the upper fuzzy distance of the α-cuts

If the distance in the consequence universe can be calculated as the difference of the coordinates, the consequence is:

$$\inf\{y_i{}^{\alpha}\} = \frac{w_{1L}^{\alpha}\inf\{B_{i,1}^{\alpha}\} + w_{2L}^{\alpha}\inf\{B_{i,2}^{\alpha}\}}{w_{1L}^{\alpha} + w_{2L}^{\alpha}}, \quad \sup\{y_i{}^{\alpha}\} = \frac{w_{1U}^{\alpha}\sup\{B_{i,1}^{\alpha}\} + w_{2U}^{\alpha}\sup\{B_{i,2}^{\alpha}\}}{w_{1U}^{\alpha} + w_{2U}^{\alpha}}$$

where

$$w^{\alpha}{}_{1L} = \frac{1}{d_L^{\alpha}(A_1,x)}, \quad w^{\alpha}{}_{2L} = \frac{1}{d_L^{\alpha}(x,A_2)}, \quad w^{\alpha}{}_{1U} = \frac{1}{d_U^{\alpha}(A_1,x)}, \quad w^{\alpha}{}_{2U} = \frac{1}{d_U^{\alpha}(x,A_2)}$$

the result: $y^{\alpha} = [\,\inf\{y^{\alpha}\}, \sup\{y^{\alpha}\}\,]$, $y = \bigcup_{\alpha} \alpha{\cdot}y^{\alpha}$ $\qquad \forall\, \alpha \in \Lambda$

The method offers real advatages if the practical calculations can be restricted to a small finite set of levels (the *important cuts*), rather than calculating for all $\alpha \in [0,1]$. In almost all practical implementations of fuzzy control, the membership functions of both the terms involved and the observation are restricted to piecewise linear, even usually trapezoidal (or sometimes triangular) shapes. We define the set of important cuts by the united breakpoint set Λ. For trapezoidal and triangular sets $\Lambda=\{0,1\}$. This means usually $2|\Lambda|$ interpolations as both the lower and the upper end (infimum and supremum) of y^α must be separately approximated, except when the cores are of zero length (e.g. the triangular case, or the sets defined in [2]), when only $2|\Lambda|-1$ interpolations are necessary.

2. The analysis of the conclusion between two breakpoint levels

Without hurting generality, in the next calculations we suppose that the two neighbouring α-levels in the breakpoint set Λ are 0 and any α (e.g. 1). All other cases can be obtained from this special case by simply scaling the flanks of the membership functions along the α axis. In this sense the left and right flanks of trapezoids are suitable for the general analysis of any piece in the piecwise linear membership function of the terms. The rules to be interpolated will be denoted by

"If x is A_1 then y is B_1" (briefly $A_1 \rightarrow B_1$) and
"If x is A_2 then y is B_2" (briefly $A_2 \rightarrow B_2$)

where x denotes the observation and y is the conclusion.

A trapezoidal membership function can be described by four points $P(a_{i1},0)$, $P(a_{i2},\alpha)$, $P(a_{i3},\alpha)$, $P(a_{i4},0)$. The examined observation x, antecedents A_i and consequents B_i pieces are trapezoidal. Then the equations of the left and right flanks of these pieces are:

$$A_i{}^{\alpha}{}_{L} = \alpha \cdot (a_{i2} - a_{i1}) + a_{i1} \quad A_i{}^{\alpha}{}_{U} = \alpha \cdot (a_{i3} - a_{i4}) + a_{i4}$$
$$B_i{}^{\alpha}{}_{L} = \alpha \cdot (b_{i2} - b_{i1}) + b_{i1} \quad B_i{}^{\alpha}{}_{U} = \alpha \cdot (b_{i3} - b_{i4}) + b_{i4}$$
$$x^{\alpha}{}_{L} = \alpha \cdot (x_2 - x_1) + x_1 \quad x^{\alpha}{}_{U} = \alpha \cdot (x_3 - x_4) + x_4$$

Statement 1. *The equation of the left slope of the conclusion calculated from the linear interpolation of the two rules $A_1 \rightarrow B_1$ and $A_2 \rightarrow B_2$ and the observation x between the two breakpoint levels 0 and α is*

$$y_L^\alpha = \frac{C_1\alpha^2 + C_2\alpha + C_3}{c_9\alpha + c_{10}}$$

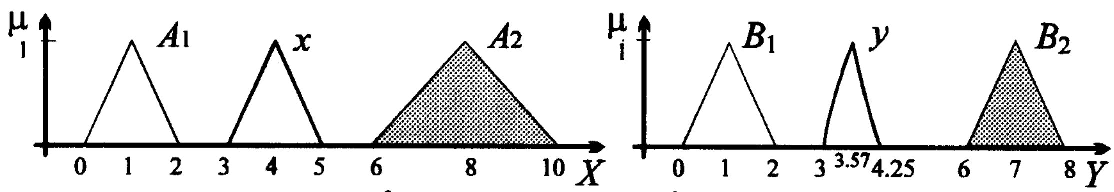
where $C_1 = c_3c_5 + c_1c_7$, $C_2 = c_3c_6 + c_4c_5 + c_1c_8 + c_2c_7$, $C_3 = c_4c_6 + c_2c_8$

$c_1 = x_2 - x_1 - a_{12} + a_{11}$, $c_2 = x_1 - a_{11}$, $c_3 = a_{22} + a_{21} - x_2 + x_1$, $c_4 = a_{21} - x_1$

$c_5 = b_{12}-b_{11}$, $c_6 = b_{11}$, $c_7 = b_{22}-b_{21}$, $c_8 = b_{21}$, $c_9 = a_{11}-a_{12}+a_{22}-a_{21}$, $c_{10} = a_{21}-a_{11}$

The right slope has a similar equation.

The result of the Statement presents that the *piecewise linearity of the membership functions* of the terms (observation, antecedents and consequents) *is not preserved in the conclusion generally* (e.g.Fig.2.).



$$y_L^\alpha = \frac{\alpha^2 + 6\alpha + 18}{\alpha + 6} \qquad y_U^\alpha = \frac{\alpha^2 - 10\alpha + 34}{-\alpha + 8}$$

Fig.2. Linearity of the membership functions of the terms
is not preserved in the conclusion

Let us examine now under which special restrictions on the shape of the terms in the rules the functions $y^{\alpha}{}_{L}$ and $y^{\alpha}{}_{U}$ become polynomial or even linear.

Corollary 1. *The flanks of y are piecewise polynomial (quadratic) if and only if the two antecedents A_1 and A_2 have equivalent piecewise linear slopes, obtainable from each other by geometric translations*

$$a_{12} - a_{11} = a_{22} - a_{21} \quad \Rightarrow \quad c_9 = 0$$

Corollary 2. *The slopes of y are piecewise linear if either both the antecedents A_i and the consequents B_i of the rules are equivalent pairwise and piecewise or if the antecedents and the observation x are all equivalent piecewise*

$$a_{12} - a_{11} = a_{22} - a_{21}\ ,\ b_{12} - b_{11} = b_{22} - b_{21} \ \Rightarrow \ c_9 = 0\ ,\ C_1 = 0$$

or $$a_{12} - a_{11} = a_{22} - a_{21} = x_2 - x_1 \quad \Rightarrow \quad c_9 = 0\ ,\ C_1 = 0$$

The conditions of preserving piecewise linearity are rather strict, they are satisfied in many practical cases, as e.g. when the state variables are covered by "equidistant" terms. Despite the importance of the special cases satisfying the conditions of Corollary 2, the result obtained is somewhat disappointing, as it indicates that in the general case interpolation only for the support and the core ($\alpha = 0,1$; or in the general case for the breakpoint set $\alpha \in \Lambda$) might not be satisfactory. If the nonlinearity of the rational function obtained for the general case is strong, it will be necessary to calculate for a much higher number of α-s and this increases the computational time.

3. The analysis of the nonlinearity of the conclusion

It is important to know, how many levels should be calculated for the interpolation, as the number of steps is proportional with the number of levels.
If the conclusion were piecewise linear (or at least practically linear), the number of the interpolation steps would be only $2|\Lambda|$. If the conclusion is far from the piecewise linearity, it is more. So it is important to check the maximal difference between the real conclusion and it's piecewise linear approximation.

Let us analyze the function of y^{α}_{L} qualitatively:
By completing the polynomial division, the function can be rewritten in the form

$$y_L^{\alpha} = \frac{C_3 c_9^2 - C_2 c_9 c_{10} + C_1 c_{10}^2}{c_9^2 (c_9 \alpha + c_{10})} + \left(\frac{C_1}{c_9} \alpha + \frac{C_2 c_9 - C_1 c_{10}}{c_9^2} \right) = \frac{\mathrm{A}}{\alpha + \mathrm{B}} + (\mathrm{C}\alpha + \mathrm{D}) = y_H + y_L$$

where

$$\mathrm{A} = \frac{C_3 c_9^2 - C_2 c_9 c_{10} + C_1 c_{10}^2}{c_9^3}\ ,\ \mathrm{B} = \frac{c_{10}}{c_9}\ ,\ \mathrm{C} = \frac{C_1}{c_9}\ ,\ \mathrm{D} = \frac{C_2 c_9 - C_1 c_{10}}{c_9^2}$$

This form shows clearly that the curve of y^{α}_{L} is the superposition of a straight line y_L, and a hyperbola y_H (see Fig. 3.).

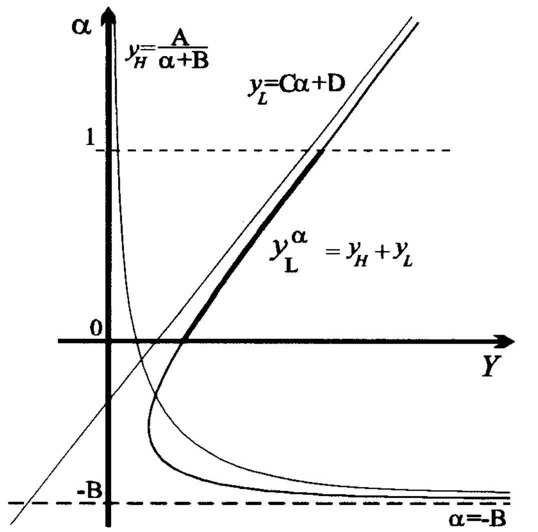
As the hyperbola is monotonicly decreasing with increasing α, the upper bound E for the linearity error can be given by calculating the difference of $y_H(0) - y_H(1)$:

$$E = y_H(0) - y_H(1) = \frac{\mathrm{A}}{\mathrm{B}(1+\mathrm{B})} = \frac{C_3 c_9^2 - C_2 c_9 c_{10} + C_1 c_{10}^2}{c_9 c_{10} (c_9 + c_{10})}$$

Statement 2. *The linearity error of y^{α}_{L} is not exceeding $\varepsilon > 0$ if*

$$\frac{(C_2 + c_{10}\varepsilon) + \sqrt{(C_2 + c_{10}\varepsilon)^2 - 4C_1(C_3 - c_{10}\varepsilon)}}{2(C_3 - c_{10}\varepsilon)} \le \frac{c_9}{c_{10}} = \frac{1}{\mathrm{B}} \le \frac{(C_2 + c_{10}\varepsilon) - \sqrt{(C_2 + c_{10}\varepsilon)^2 - 4C_1(C_3 - c_{10}\varepsilon)}}{2(C_3 - c_{10}\varepsilon)}$$

It seems to be reasonable to express the condition for $c_9 = a_{11}-a_{12}+a_{22}-a_{21}$ in a form relative to $c_{10} = a_{21}-a_{11}$, as in this form it is much clearer that the difference of the degree of fuzziness between the two antecedents (measured by the unit of their distance from each other) must be limited. For details of the proof see e.g. [9,10].



This latter has a horizontal asymptote at α = -B and a vertical one at y = 0. The combination has the same horizontal asymptote and another one at the linear component of the function. The left slope of the conclusion $y^{\alpha}{}_{L}$ is given by the section of the curve from $\alpha = 0$ to $\alpha = 1$. If $\alpha \to \pm\infty$, the curve converges to its asymptote (for large α-s it is approximately linear). The larger is B, the farther is $\alpha = 0$ from the neighborhood of the focus, i.e. the area where the curve is very nonlinear.

Fig.3. $y^{\alpha}{}_{L}$ is the superposition of straight line y_L , and hyperbola y_H

Summarizing our results we can state that in some special but practically important cases the slopes of the calculated conclusion are exactly linear, while in the general case hyperbolic, however, in most cases rather close to linear so that in the case of piecewise linear rules and observation *it is sufficient to calculate only the breakpoints* Λ *of* y, i.e. in the trapezoidal or triangular case for $\Lambda = \{0,1\}$.

4. The convexity and normality of the fuzzy conclusion

The terms and the observation of the classical reasoning algorithms [1,2], and the linear rule interpolation are restricted to convex and normal fuzzy sets. Using multiple level reasoning it is important to check the normality of the generated results between the reasoning levels. Therefore, it is an interesting question to examine if the conclusion generated by the linear rule interpolation is suitable for further reasoning steps; if it keeps normality or not.

Let us check first the convexity. The conclusion y is convex if all its α-cuts are connected. The interpolation method applied never produces other than connected cuts as they are expressedly defined as intervals, by their minimum and maximum. So convexity is automatically satisfied.

It is much more complicated to overview the situation from the point of view of normality. The conclusion is normal, if the membership function of y assumes all values in [0,1]. Obviously, the method of generating y is such that the two points for the intervals should be always ordered (in the sense of () in the proper way, i.e.

$$\forall \alpha : \inf\{y^{\alpha}\} \leq \sup\{y^{\alpha}\}$$

If the above condition is not fulfilled, the "membership function" wraps around itself (forms a loop (e.g.Fig.4.)). For any α where this condition is not satisfied, a real membership function does not exist, i.e. $\text{height}(y) = \max_{[0,1]}\{\alpha \mid \inf(y^{\alpha}) \leq \sup(y^{\alpha})\}$

Using the equations and denotation of the previous sections, we can formulate the condition of normality:

Statement 3. *The conclusion y is normal if and only if*

$$y_2 = \frac{(a_{22}-x_2)b_{12}+(x_2-a_{12})b_{22}}{a_{22}-a_{12}} \le y_3 = \frac{(a_{23}-x_3)b_{13}+(x_3-a_{13})b_{23}}{a_{23}-a_{13}}$$

With introducing new denotations for all core lengths and the distances between the cores of the neighboring sets:

$$K_{a1} = a_{13} - a_{12}\,,\ K_{a2} = a_{23} - a_{22}\,,\ K_x = x_3 - x_2\,,\ K_{b1} = b_{13} - b_{12}\,,\ K_{b2} = b_{23} - b_{22}$$
$$d_{a1} = x_2 - a_{13}\,,\ d_{a2} = a_{22} - x_3\,,\ d_b = b_{22} - b_{13}$$

the formulation of statement 3 is:

$$d_b[(K_{a1}+d_{a1})(K_{a2}+d_{a2})-(K_x+d_{a1})(K_x+d_{a2})] \le$$
$$\le (K_{a1}+d_{a1})(d_{a1}+K_x)K_{b2}+(K_{a2}+d_{a2})(d_{a2}+K_x)K_{b1}$$

Corollary 3. *If the rules and the observation contain only triangular membership functions, the conclusion is always normal.* ($K_{ai} = K_{bi} = K_x = 0$) (e.g.Fig.2.)

Corollary 4. *If the corresponding cores in the rules have uniform length, the conclusion will be normal if and only if* ($K_{ai} = K_a$, $K_{bi} = K_b$)

$$d_b(K_a - K_x) \le K_b(d_{a1}+d_{a2}+2K_x)$$

In the worst case $K_x = 0$

$$d_b(K_a - K_x) \le K_b d_a\ ,\ \text{where } d_a = a_{22} - a_{13}$$

Corollary 5. *If* $K_{ai} = K_x$, $K_{bi} = K_b$, *the conclusion is always normal.* (e.g.Fig.5.)

Corollary 6. *In the case of uniform core length in both the antecedent and the consequent parts of the rules, the conclusion will always be normal if the ratio of the distances of the rule cores of the consequents and of the antecedents does not exceed the ratio of the core lengths themselves.*

If the scale at both universes is locally normalized by the distance between the two rule cores (obtaining the normalized cores $k_a=K_a/d_a$, $k_x=K_x/d_a$, $k_b=K_b/d_b$), the conclusion will always be normal if the consequents have a not shorter core, i.e. *the consequents are not less fuzzy then the antecedents* themselves. (The opposite is not true: if the consequents are less fuzzy, still a fuzzy enough observation might save the normality of the conclusion.) In the normalized scale we have $k_a \le k_b + k_x$ as a sufficient (but still not necessary) condition. It is enough if the observation is at least as fuzzy as the antecedents for guaranteeing the normality of the conclusion.

5. Conclusions

In this paper it was examined whether the piecewise linearity of all membership functions in an If...then rule model and corresponding observation is preserved for the conclusion constructed by α-cut interpolation for the breakpoint levels. We have shown that in general, the pieces of the generated conclusion are not linear, even not polynomial, although they are very close to linear in the domain [0,1]. It was investigated under what conditions the rational function reduces to polynomial (quadratic) and to linear (important special cases). The shape of the obtained rational function was analyzed qualitatively and it turned out to be essentially hyperbolic. If the part of the hyperbolic curve corresponding to the slope of the conclusion is far enough from the intensively curved "central" area (close to the focus point), the degree of nonlinearity is low (Fig.3.). The results obtained that practiceally in almost all cases it is

enough to interpolate for the breakpoint levels only. This provides low computational time for the algorithm.
It was also examined if the conclusion maintained convexity and normality. Convexity automatically follows from the algorithm, but the normality is not always true. Moreover there are cases - because of the non-normality - when the conclusion does not exist at all. Conditions for normality in general (with some special cases) and for the reality of the generated conclusion were also determined.

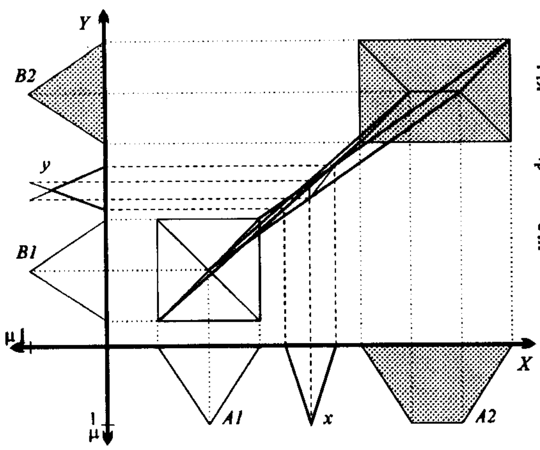


Fig.4. Normality of the conclusion is not always satisfied

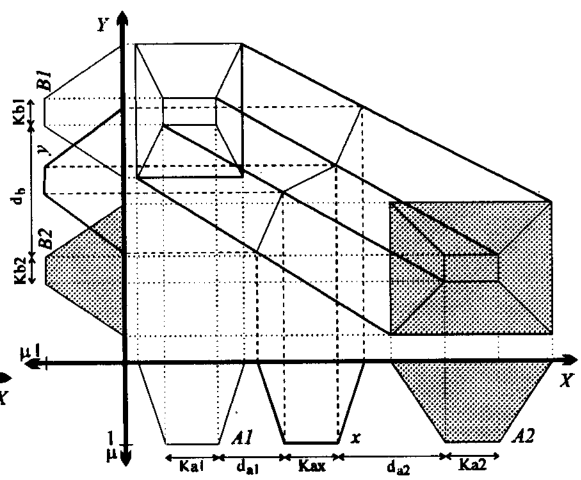


Fig.5. If $K_{ai} = K_x$, $K_{bi} = K_b$, the conclusion is always normal

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