NONLINEAR DIFFERENCE EQUATIONS AND NEURAL NETS

Endre Pap
Institute of Mathematics, University of Novi Sad
Trg D. Obradovića 4, 21000 Novi Sad, Yugoslavia

1. Introduction

We shall examine the limit behavior of some difference equations which models some neural nets. The problem of the construction of the computers with parallel processinngs (neural nets) is an optimization problem in the following sense. If we denote by K the number of elementary processors (neurons) then the complexity of the algorithm grows with the number K, in the worst case as K^2. So it is important to examine the limit case $K \to \infty$.
We shall examine the limit behaviour of some difference equatios which models some neural nets. But for most of considered problems the limit equation (obtained as limit of difference equations) has no differentiable solution, since the initial condition usually gives this. For that purpose we shall use decomposable measures and the corresponding integrals ([2],[4],[7],[8],[9],[10],[11],

,[16], [17]).

2. Preliminaries

Let $[a, b]$ be a closed (in some cases semiclosed) subinterval of $[-\infty, +\infty]$. We shall consider a partial order $\leq$ on $[a, b]$, which can be the usual order of the real line, but it can also be another order. All future considerations will be with respect to the order $\leq$.

The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a, b] \times [a, b] \to [a, b]$ which is commutative, nondecreasing (with respect to $\leq$) associative and either a or b is a zero element, denoted by 0, i.e. for each $x \in [a, b]$

$$0 \oplus x = x \quad \text{holds.}$$

Let $[a,b]_+ = \{x : x \in [a,b],\ \ x \geq 0\}$.

The operation $\otimes$ (pseudo-multiplication) is a function $\otimes : [a,b] \times [a,b] \to [a,b]$ which is commutative, positively nondecreasing, i.e. $x \leq y$ implies $x \otimes z \leq y \otimes z$, $z \in [a,b]_+$, associative and for which there exist a unit element $1 \in [a,b]$, i.e. for each $x \in [a,b]$

$$1 \otimes x = x.$$

We suppose, further, $0 \otimes x = 0$ and that $\otimes$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e.

$$x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z).$$

Pseudo-addition $\oplus$ is idempotent if for any $x \in [a,b]$

$$x \oplus x = x \quad \text{holds.}$$

Let X be a non-empty set. Let $\sum$ be a $\sigma-$algebra of subsets of X.

A set function $m : \sum \to [a,b]_+$ (or semiclosed interval) is a $\bigoplus-$decomposable measure if there hold

$$m(\emptyset) = 0 \quad \text{(if } \oplus \text{ is not idempotent)}$$

$$m(A \cup B) = m(A) \oplus m(B)$$

for $A, B \in \sum$ such that $A \cap B = \emptyset$.
In the case when $\oplus$ is idempotent, it is possible that m is not defined on an empty set.

A $\bigoplus-$decomposable measure m is $\sigma\bigoplus-$decomposable if

$$m(\bigcup_{i=1}^{\infty} A_i) = \bigoplus_{i=1}^{\infty} m(A_i)$$

hold for any sequence (A_i) of pairwise disjoint sets from $\sum$.

Let m be a σ $\bigoplus-$decomposable measure. A function $f : X \to [a,b]$ is measurable from below if for any $c \in [a,b]$ the sets $\{x : f(x) \leq c\}$ and

$\{x : f(x) < c\}$ belong to $\sum$. f is measurable, if it is measurable from below and the sets $\{x : f(x) \geq c\}$ and $\{x : f(x) > c\}$ belong to $\sum$.

Let f and g be two functions defined on X and with values in $[a, b]$. Then, we define for any $x \in X$

$$(f \oplus g)(x) = f(x) \oplus g(x) \ ,$$

$$(f \otimes g)(x) = f(x) \otimes g(x)$$

and for any $c \in [a, b]$

$$(c \otimes f)(x) = c \otimes f(x).$$

We suppose further that $([a, b], \oplus)$ and $([a, b], \otimes)$ are complete lattice ordered semigroups. A complete lattice means that for each set $A \subset [a, b]$ bounded from above (below) there exists $\sup A$ ($\inf A$). Further, we suppose that $[a, b]$ is endowed with a metric d compatible with sup and inf and which satisfies at laest one of the following conditions:

(a) $d(x \oplus y,\ x' \oplus y') \leq d(x, x') + d(y, y')$

(b) $d(x \oplus y,\ x' \oplus y') \leq \max\{d(x, x'),\ d(y, y')\}$.

Both conditions (a) and (b) imply that :

$$d(x_n, y_n) \to 0 \quad \text{implies} \quad d(x_n \oplus z,\ y_n \oplus z) \to 0.$$

Condition (b) implies

$$d(\bigoplus_{i=1}^{n} x_i,\ \bigoplus_{j=1}^{n} y_j) \leq \min_j \max_i d(x_i, y_j).$$

We suppose further the monotonicity of the metric d, i.e.

$$x \leq z \leq y \quad \text{implies} \quad d(x, y) \geq \max\{d(y, z), d(x, z)\}.$$

Let ε be a positive real number, and $B \subset [a, b]$. A subset $\{l_i^\varepsilon\}$ is a $\varepsilon-$net if for each $x \in B$ there exists l_i^ε such that $d(l_i^\varepsilon, x) \leq \varepsilon$. If we have $l_i^\varepsilon \leq x$, then we shall call $\{l_i^\varepsilon\}$ a <u>lower</u> $\varepsilon-$net. If $l_i^\varepsilon \leq l_{i+1}^\varepsilon$ holds, then $\{l_i^\varepsilon\}$ is <u>monotone</u>.

We define the characteristic function

$$\chi_A(x) = \begin{cases} 0, & x \notin A \\ 1, & x \in A \end{cases} .$$

A mapping $e : X \to [a,b]$ is an elementary (measurable) function if it has the following representation

$$e = \bigoplus_{i=1}^{\infty} a_i \otimes \chi_{A_i} \quad \text{for} \quad a_i \in [a,b]$$

and $A_i \in \sum$ disjoint if $\oplus$ is not idempotent.

Definition 1. *The integral of a simple function $s = \bigoplus_{i=1}^{n} a_i \otimes \chi_{A_i}$ for $a_i \in [a,b]$ with disjoint $A_1, A_2, \ldots A_n$, if $\oplus$ is not idempotent, is defined by*

$$\int_X^{\oplus} s \otimes dm := \bigoplus_{i=1}^{n} a_i \otimes m(A_i).$$

The integral of an elementary function

$$e = \bigoplus_{i=1}^{\infty} a_i \otimes \chi_{A_i} \quad \textit{for} \quad a_i \in [a,b] \ \ (i \in N) \ \ \textit{with} \ (A_i)$$

disjoint if $\oplus$ is not idempotent, is defined by

$$\int_X^{\oplus} e \otimes dm := \bigoplus_{i=1}^{\infty} a_i \otimes m(A_i).$$

The integral of a bounded measurable (from below for $\oplus$ idempotent) function $f : X \to [a,b]$, for which, if $\oplus$ is not idempotent for each $\varepsilon > 0$, there exists a monotone $\varepsilon-$net in $f(X)$, is defined by

$$\int_X^{\oplus} f \otimes dm := \lim_{n\to\infty} \int_X^{\oplus} \varphi_n(x) \otimes dm,$$

(where (φ_n) is the sequence of elementary functions constructed in Theorem 1. in [15]).

3. Nonlinear difference equation

We shall consider for given $\alpha, \beta \in [a, b]$ the following difference equation

$$c_{m,n}^{k+1} = \alpha \otimes c_{m-1,n}^{k} \oplus \beta \otimes c_{m,n-1}^{k} \tag{1}$$

where $k = 0, 1, 2, ...;\ m, n = 0, \pm 1, \pm 2, ...$
with the initial condition

$$c_{m,n}^{0} = \begin{cases} 1, & n = 0,\ m \geq 0 \\ 1, & m = 0,\ n \geq 0 \\ \mathbf{0}, & \text{otherwise} . \end{cases} \tag{2}$$

Example 1. Let $\oplus = \min$ and $\otimes = +$ on $[-\infty, +\infty]$. Then we have $\mathbf{0} = +\infty$ and $1 = 0$. Taking $\alpha = \beta = 1 = 0$ the equation (1) reduces on the Bellman equation (which appears for example in the construction of parallel processor of multiplication of matrices)

$$c_{m,n}^{k+1} = \min\{c_{m-1,n}^{k}, c_{m,n-1}^{k}\}$$

where $k = 0, 1, 2, ...;\ m, n = 0, \pm 1, \pm 2, ...,$

with the initial conditions

$$c_{m,n}^{0} = \begin{cases} 0, & n = 0,\ m \geq 0 \\ 0, & m = 0,\ n \geq 0 \\ +\infty, & \textbf{otherwise} . \end{cases}$$

We shall introduce the operator $T : C^2 \to C$, where

$$C = \{c_{m,n}^{k} : k = 0, 1, 2...; m, n = 0, \pm 1, \pm 2, ...\}$$

and T acts in the following way

$$Tc_{m,n}^{k} = \alpha \otimes c_{m-1,n}^{k} \oplus \beta \otimes c_{m,n-1}^{k}.$$

This operator is linear with respect to $\oplus$ and $\otimes$.

4. Limit behavior

Now we shall consider the corresponding continuous analog of the problem (1) – (2).
We consider functions defined on $[0, M] \times [-\infty, +\infty]^2$, where $M > 0$, and with values in $[a, b]$, i.e., $a : [0, M] \times [-\infty, +\infty]^2 \to [a, b]$. Then the corresponding equation to (1) of continuous variable with mesh size $h > 0$ has the form

$$(3) \qquad c_h(z + h, x, y) = \alpha \otimes c_h(z, x - h, y) \oplus \beta \otimes c_h(z, x, y - h),$$

taking $z = kh, \ k = 0, 1, 2, \ldots$ and $c(kh, mh, nh) = c^k_{m,n}$.
The corresponding initial condition to (2) has the form

$$(4) \qquad c_h(0, x, y) = c^0(x, y) = \begin{cases} 1, & y = 0, \ x \geq 0 \\ 1, & x = 0, \ y \geq 0 \\ 0, & \text{otherwise} \ . \end{cases}$$

Introducing the operator $T_h : C_h^2 \to C_h$, where

$$C_h = \{c_h : c_h(kh, mh, nh) = c^k_{m,n} \ \text{for} k = 0, 1, 2, \ldots; \ m, n = 0, \pm 1, \pm 2, \ldots\},$$

such that

$$T_h c_h(z, x, y) = \alpha \otimes c_h(z, x - h, y) \oplus \beta \otimes c_h(z, x, y - h)$$

for $z = kh$ $(k = 0, 1, 2, \ldots)$, we obtain that the solution of the problem $(3) - (4)$ is given by

$$(5) \qquad c_h(z, x, y) = (T_h)^k c^0(x, y).$$

We introduce the pseudo-scalar product for functions with values in $[a, b]$

$$(f, g)_\oplus = \int^\oplus f(x, y) \otimes g(x, y) dx dy.$$

We take $k \to \infty$ or $h \to 0$ (since the interval $[0, M]$ is bounded it is the same) for a point $z_0 \in [0, M]$ that $kh \to z_0$. We have for a smooth function $f(x, y)$

$$(c_h(z_0, x, y), f(x, y))_\oplus = ((T_h)^k c^0(x, y), f(x, y))_\oplus = (c^0(x, y), (T_h^*)^k f(x, y))_\oplus,$$

where T_h^* is the adjoint operator of the operator T_h with respect to the pseudo-scalar product $(.,.)_\oplus$.
The weak $-\oplus-$limit of solutions $c_h(z_0, x, y)$ of the problem $(3)-(4)$,since

$$\lim_{h\to 0}(c(z_0, x, y), f(x, y))_\oplus = F(f(x, y))$$

is a pseudo-linear functional, we shall denote by $c(z_0, x, y)$, i.e., $F(f(x, y)) = (c(z_0, x, y), f(x, y))_\oplus$.

Example 2. *Take Min - Plus on the interval* $[-\infty, +\infty]$. *Then we have for the solution* (5) *of the problem* $(3)-(4)$

$$c_h(z, x, y) = (T_h)^k c^0(x, y) = \min_{r_1+r_2=k,\ r_i\in\{0,1,2,\ldots\}}\{r_1\alpha + r_2\beta + c^0(x - r_1 h, y - r_2 h)\}.$$

The adjoint operator $(T_h^*)^k$ *acts in the following way*

$$(T_h^*)^k f(x, y) = \min_{r_1+r_2=k,\ r_i\in\{0,1,2,\ldots\}}\{-r_1\alpha - r_2\beta + c^0(x + r_1 h, y + r_2 h)\}.$$

Taking the weak$-\oplus-$*limit we obtain that*

$$(c(z_0, x, y), f(x, y))_\oplus = (c^0(x, y), \min_{r_1+r_2=z_0,\ r_i\in[0,+\infty)}\{-r_1\alpha - r_2\beta + f(x + r_1, y + r_2)\}.$$

The limit equation to (3) *on smooth function is*

$$\frac{\partial c}{\partial z} = \min\{\alpha - \frac{\partial c}{\partial x}, \beta - \frac{\partial c}{\partial y}\},$$

whose solution can be find by the Pontrjagin maximum principe .

References

[1] J. Aczel, Lectures on Functional Equations and their Applications, Academic Press, New York,1966.

[2] D. Dubois, M. Prade, Fuzzy sets and systems: Theory and applications, Academic Press, New York, 1980

[3] D.Dubois, M.Prade, A class of fuzzy measures based on triangular norms, Internat. J.Gen. System 8 (1982), 43-61.

[4] H. Ichihashi, M. Tanaka, K. Asai, Fuzzy Integrals Based on Pseudo-Additions and Multiplications, J. Math. Anal. Appl. 130 (1988), 354-364.

[5] C.M. Ling, Representation of associative functions, Publ. Math. Debrecen 12 (1965), 189 -212.

[6] V. Maslov, S.N. Samborski, Idempotent Analysis, Advances in Soviet Mathematics 13, Amer. Math. Soc., Providence, 1992.

[7] R. Mesiar, J. Rybarik, Pseudo-arithmetical operations, Tatra Mountains 2 (1993), 185-192.

[8] T.Murofushi, M.Sugeno, Pseudo - additive measures and integrals, J.Math.Anal.Appl. 122 (1987),197-222.

[9] E.Pap, On non-additive set functions, Atti. Sem. Mat. Fis. Univ. Modena 39 (1991), 345-360.

[10] E.Pap, Lebesgue and Saks decompositions of $\perp$ - decomposable measures, Fuzzy Sets and Systems 38 (1990), 345-353.

[11] E.Pap, Extension of the continuous t-conorm decomposable measure,Univ. u Novom Sadu Zb. Rad. Prirod.-Mat.Fak. Ser. Mat. 20,2 (1990),121-130.

[12] E.Pap, Solution of nonlinear differential and difference equations, EUFIT 93, Aachen, 498-503.

[13] E. Pap, $g-$calculus, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. (to appear), Preprint 1991.

[14] E.Pap, Decomposable measures and applications on nonlinear partial differential equations, Rend. Mat. del Palermo ser II - num 28 (1992), 387-403.

[15] E.Pap, Integral generated by decomposable measure, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 20,1 (1990), 135-144.

[16] E.Pap, Null-additive Set Functions, Kluwer Academic Publ. , Ister Science Press (to appear).

[17] S.Weber, $\perp$ - decomposable measures and integrals for Archimedean t-conorm, J. Math. Anal. Appl. 101 (1984), 114-138.