RELATIONS BETWEEN THE QUANTORS AND THE MODAL TYPE OF OPERATORS IN INTUITIONISTIC FUZZY LOGICS

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Following the ideas and using the notation from [1-3], we shall introduce the basic relations between the quantors (see [4,5]) and modal type of operators (see [1-3,5,6]) in the Intuitionistic Fuzzy Modal Logic (IFML).

Initially, following [4], we shall define the quantors "∀" and "∃".

Let us assume that the language is without functional symbols (for simplicity of presentation), i.e. atomic formulae are of the kind $P(x_1, x_2, \ldots, x_n)$, where P is an n-ary predicate symbol, x_1, x_2, ..., x_n are n individual variables. Predicate logic formulae are built up from atomic formulae by means of the propositional operations "&", "∨", "⊃", "¬", "≡" and by application of quantifiers, i.e., if A is a formula, x - a variable, then ∀xA and ∃xA are formulae.

Truth values of predicate formulae are obtained, if a domain of interpretation E is fixed, usually called the universe of the interpretation. Atomic formulae get their meaning through interpretation functions i which assign to each variable x an element $i(x) \in E$. The truth value of a given atomic formula $P(x_1, x_2, \ldots, x_n)$ under the interpretation function i is determined by an evaluation function V which assigns to each n-ary predicate symbol P a function $V(P): E^n \rightarrow [0, 1] \times [0, 1]$. The pair (E, V) is called a model. In this situation we have (for a given i):

$$V(P(x_1, x_2, \ldots, x_n)) = V(P)(i(x_1), i(x_2), \ldots, i(x_n)).$$

The evaluation V can be extended for arbitrary formulae by the inductive clauses for "&", "∨", "⊃", "¬", "≡".

The definition for the quantifiers is as follows:

$$V(\forall xA) = \langle \min_{a\in E} \mu(A(i(x)=a), \max_{a\in E} \gamma(A(i(x)=a)\rangle$$

$$V(\exists xA) = \langle \max_{a\in E} \mu(A(i(x)=a), \min_{a\in E} \gamma(A(i(x)=a)\rangle$$

which can be denoted simpler (where "x ranges over E"):

$$V(\forall xA) = \langle \min_{x} \mu(A), \max_{x} \gamma(A)\rangle$$

$$V(\exists xA) = \langle \max_{x} \mu(A), \min_{x} \gamma(A)\rangle.$$

For the formula A and for the variable x, by analogy with the operations over IF logic (see [7]), it will be convenient to define the following:

$$V(\forall xA) = \forall xV(A)$$

$$V(\exists xA) = \exists xV(A).$$

Predicate IFTs can be defined just as their propositional counterparts: these are the formulae which get the valuation with $\mu \geq \gamma$ for every model and interpretation. The logical axioms of the theory K (see [8]):

$A \supset (B \supset A)$,

$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$,

$(\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)$

$\forall xA(x) \supset A(t)$, for the fixed variable t,

$\forall x(A \supset B) \supset (A \supset \forall xB)$,

are IFTs (see [4]). The same is also valid for

$(\forall xA(x) \supset B) \equiv \exists x(A(x) \supset B)$,

$\exists xA(x) \supset B \equiv \forall x(A(x) \supset B)$,

$B \supset \forall xA(x) \equiv \forall x(B \supset A(x))$,

$B \supset \exists xA(x) \equiv \exists x(B \supset A(x))$,

$(\forall xA \ \& \ \forall xB) \equiv \forall x(A \ \& \ B)$,

$(\forall xA \vee \forall xB) \supset \forall x(A \vee B)$,

$\neg\forall xA \equiv \exists x\neg A$,

$\neg\exists xA \equiv \forall x\neg A$,

$\forall x\forall yA \equiv \forall y\forall xA$,

$\exists x\exists yA \equiv \exists y\exists xA$,

$\exists x\forall yA \supset \forall y\exists xA$

$\forall x(A \supset B) \supset (\forall xA \supset \forall xB)$, The link between the interpretations of quantifiers and the topological operators C (closure) and I (interior) defined over IFS [9] is obvious. Thus, the equalities from [10] can be transformed for the IFML. At the result, we can prove the following

THEOREM 1: Let A be a formula and x be a variable. Then

(a) $V(\forall x\Box A) = V(\Box\forall xA)$,

(b) $V(\exists x\Box A) = V(\Box\exists xA)$,

(c) $V(\forall x\Diamond A) = V(\Diamond\forall xA)$,

(d) $V(\exists x\Diamond A) = V(\Diamond\exists xA)$.

Proof: (a) $V(\forall x\Box A) = \langle\min\limits_x \mu(A),\ \max\limits_x 1 - \mu(A)\rangle$

$= \langle\min\limits_x \mu(A),\ 1 - \min\limits_x \mu(A)\rangle = V(\Box\forall xA)$.

THEOREM 2: Let A be a formula and x be a variable. Then

(a) $V(\Box\exists x\Box A) = V(\Diamond\exists x\Box A) = V(\overline{\Box\forall x\Diamond\bar{A}}) = V(\overline{\Diamond\forall x\Diamond\bar{A}})$

(b) $V(\Box\exists x\Diamond A) = V(\Diamond\exists x\Diamond A) = V(\overline{\Box\forall x\Box\bar{A}}) = V(\overline{\Diamond\forall x\Box\bar{A}})$

(c) $V(\Box\forall x\Box A) = V(\Diamond\forall x\Box A) = V(\overline{\Box\exists x\Diamond\bar{A}}) = V(\overline{\Diamond\exists x\Diamond\bar{A}})$

(d) $V(\Box\forall x\Diamond A) = V(\Diamond\forall x\Diamond A) = V(\overline{\Box\exists x\Box\bar{A}}) = V(\overline{\Diamond\exists x\Box\bar{A}})$

(e) $V(\Box\exists x\Box\bar{A}) = V(\Diamond\exists x\Box\bar{A}) = V(\overline{\Box\forall x\Diamond A}) = V(\overline{\Diamond\forall x\Diamond A})$

(f) $V(\Box\exists x\Diamond\bar{A}) = V(\Diamond\exists x\Diamond\bar{A}) = V(\overline{\Box\forall x\Box A}) = V(\overline{\Diamond\forall x\Box A})$

(g) $V(\Box\forall x\Box\bar{A}) = V(\Diamond\forall x\Box\bar{A}) = V(\overline{\Box\exists x\Diamond A}) = V(\overline{\Diamond\exists x\Diamond A})$

(h) $V(\Box\forall x\Diamond\bar{A}) = V(\Diamond\forall x\Diamond\bar{A}) = V(\overline{\Box\exists x\Box A}) = V(\overline{\Diamond\exists x\Box A})$

Proof: (a) $V(\Box\exists x\Box A) = \Box\exists x\Box V(A) = \Box\exists x\langle\mu(A),\ 1-\mu(A)\rangle$

$= \Box\langle\max\limits_x \mu(A),\ \min\limits_x(1-\mu(A))\rangle = \langle\max\limits_x \mu(A),\ 1 - \max\limits_x \mu(A)\rangle$;

$V(\Diamond\exists x\Box A) = \Diamond\langle\max\limits_x \mu(A),\ \min\limits_x(1-\mu(A))\rangle$

$= \langle 1-\min\limits_x(1-\mu(A)),\ \min\limits_x(1-\mu(A))\rangle = \langle\max\limits_x \mu(A),\ 1-\max\limits_x \mu(A)\rangle$;

$V(\overline{\Box\forall x\Diamond\bar{A}}) = \overline{\Box\forall x\Diamond\langle\gamma(A),\ \mu(A)\rangle} = \overline{\Box\forall x\langle 1-\mu(A),\ \mu(A)\rangle}$

$= \overline{\Box\langle\min\limits_x(1-\mu(A)),\ \max\limits_x \mu(A)\rangle} = \overline{\langle\min\limits_x(1-\mu(A)),\ 1 - \min\limits_x(1 - \mu(A)\rangle}$

$= \langle 1 - \min\limits_x(1-\mu(A)),\ \min\limits_x(1 - \mu(A)\rangle = \langle\max\limits_x \mu(A),\ 1-\max\limits_x \mu(A)\rangle$;

$V(\overline{\Diamond\forall x\Diamond\overline{A}}) = \overline{\Diamond\langle \min_x(1-\mu(A)), \max_x \mu(A)\rangle} = \overline{\langle 1 - \max_x \mu(A), \max_x \mu(A)\rangle}$

$= \langle \max_x \mu(A), 1-\max_x \mu(A)\rangle.$

(e) $V(\Box\exists x\Box\overline{A}) = V(\overline{\overline{\Box\exists x\Box\overline{A}}}) = \overline{\langle 1 - \min_x \gamma(A), \min_x \gamma(A)\rangle}$

$= \langle \min_x \gamma(A), 1 - \min_x \gamma(A)\rangle$

e.t.c.

All other assertions are proved analogically.

Let for a fixed formula A and for a variable x:

$S(A) = \{\Box\exists x\Box A, \Diamond\exists x\Box A, \overline{\Box\forall x\Diamond\overline{A}}, \overline{\Diamond\forall x\Diamond\overline{A}}\},$

$T(A) = \{\Box\exists x\Diamond A, \Diamond\exists x\Diamond A, \overline{\Box\forall x\Box\overline{A}}, \overline{\Diamond\forall x\Box\overline{A}}\},$

$U(A) = \{\Box\forall x\Box A, \Diamond\forall x\Box A, \overline{\Box\forall x\Diamond\overline{A}}, \overline{\Diamond\forall x\Diamond\overline{A}}\},$

$V(A) = \{\Box\forall x\Diamond A, \Diamond\forall x\Diamond A, \overline{\Box\exists x\Box\overline{A}}, \overline{\Diamond\exists x\Box\overline{A}}\},$

$W(A) = \{\Box\exists x\Box\overline{A}, \Diamond\exists x\Box\overline{A}, \overline{\Box\forall x\Diamond A}, \overline{\Diamond\forall x\Diamond A}\},$

$X(A) = \{\Box\exists x\Diamond\overline{A}, \Diamond\exists x\Diamond\overline{A}, \overline{\Box\forall x\Box A}, \overline{\Diamond\forall x\Box A}\},$

$Y(A) = \{\Box\forall x\Box\overline{A}, \Diamond\forall x\Box\overline{A}, \overline{\Box\exists x\Diamond A}, \overline{\Diamond\exists x\Diamond A}\},$

$Z(A) = \{\Box\forall x\Diamond\overline{A}, \Diamond\forall x\Diamond\overline{A}, \overline{\Box\exists x\Box A}, \overline{\Diamond\exists x\Box A}\}.$

THEOREM 3: Let A be a formula and x be a variable. Then:

(a) if $P\in S(A)$ and $Q\in T(A)$, then $V(P) \le V(\exists xA) \le V(Q)$,

(b) if $P\in U(A)$ and $Q\in V(A)$, then $V(P) \le V(\forall xA) \le V(Q)$,

(c) if $P\in W(A)$ and $Q\in X(A)$, then $V(P) \le V(\forall xA) \le V(Q)$,

(d) if $P\in Y(A)$ and $Q\in Z(A)$, then $V(P) \le V(\exists xA) \le V(Q)$,

where $V(X) \le V(Y)$ for the formulae X and Y iff $\mu(X) \le \mu(Y)$ and $\gamma(X) \ge \gamma(Y)$.

THEOREM 4: Let A be a formula and x be a variable. Then for every $\alpha, \beta \in [0, 1]$:

$V(\forall x G_{\alpha,\beta}(A)) = V(G_{\alpha,\beta}(\forall xA))$

THEOREM 5: Let A be a formula and x be a variable. Then for every $\alpha, \beta \in [0, 1]$, such that $\alpha + \beta \le 1$:

(a) $V(\exists x P_{\alpha,\beta}(A)) = V(P_{\alpha,\beta}(\exists x A))$,

(b) $V(\forall x Q_{\alpha,\beta}(A)) = V(Q_{\alpha,\beta}(\forall x A))$.

Proof: (a) $V(\exists x P_{\alpha,\beta}(A)) = \exists x \langle \max(\alpha, \mu(A)), \min(\beta, \gamma(A))\rangle$

$= \langle \max_x(\max(\alpha, \mu(A))), \min_x(\min(\beta, \gamma(A)))\rangle$

$= \langle x, \max(\alpha, \max_x \mu(A)), \min(\beta, \min_x \gamma(A))\rangle = V(P_{\alpha,\beta}(\exists x A))$.

The other relations having the above form between quantors and modal type of operators are not valid.

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