NORMS AND METRICS OVER INTUITIONISTIC FUZZY LOGICS

Krassimir T. Atanassov
Math. Research Lab. - IPACT, P.O.Box 12, Sofia-1113, BULGARIA

Let us emphasize in the first place (similarly to [1]) that here we do not study these properties of the Intuitionistic Fuzzy Logics (IFLs) (see [2,3]) which follow directly from the fact that the support of an IFL is a set in the sense of the set theory (see [4]).

Here we shall use a metric, which is not related to the elements of a fixed set of propositions S and to the values of the function V, i.e. of the functions μ and γ, defined for these elements.

This peculiarity is based on the fact, that the "norm" of a given IFL's element is really not a norm in the sense of [5,6]. This "norm" is in some sense a "pseudo-norm", which assigns a number to the element $p \in S$. This number is related with the values of the functions μ and γ (which are calculated for this element). Thus the important conditions:

$$\|p\| = 0 \text{ iff } p = 0,$$

and

$$\|p\| = \|q\| \text{ iff } p = q,$$

are not valid here. Instead the following ones are valid:

$$\|p\| = \|q\| \text{ iff } \mu(p) = \mu(q) \ \& \ \gamma(p) = \gamma(q).$$

Really, the value of $\mu(p)$ plays the role of a norm (more precise - pseudo-norm) for the element $p \in S$.

In the case of the intuitionistic fuzzing, the existing of the second functional component - the function γ - generates different possibilities for the defining of the concept "norm" (in the sense of pseudo-norm) over S.

The first norm for every propositional form A (c.f. [7]: each proposition is a propositional form; if A is a propositional form then $\neg A$ is a propositional form; if A and B are propositional forms, then A & B, $A \vee B$, $A \supset B$ are propositional forms) is:

$$\sigma(A) = \mu(A) + \gamma(A).$$

It assigns the degree of "definiteness" of A. From

$$\pi(A) = 1 - \mu(A) - \gamma(A)$$

it follows that

$$\sigma(A) = 1 - \pi(A).$$

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THEOREM 1: For every two propositional forms A and B (see [8,9]):

(a) $\sigma(\neg A) = \sigma(A)$;

(b) $\sigma(A \& B) \geq \min(\sigma(A), \sigma(B))$;

(c) $\sigma(A \veebar B) \leq \max(\sigma(A), \sigma(B))$;

(d) $\sigma(A \supset B) \leq 1$, for the two types of the implication;

(e) $\sigma(\exists x A(x)) \geq \max_{x \in S} \sigma(A(x))$;

(f) $\sigma(\forall x A(x)) \leq \min_{x \in S} \sigma(A(x))$;

(g) $\sigma(\Box A) = 1$;

(h) $\sigma(\Diamond A) = 1$;

(i) $\sigma(D_{\alpha}(A)) = 1$, for every $\alpha \in [0, 1]$;

(j) $\sigma(F_{\alpha,\beta}(A)) = \alpha + \beta + (1 - \alpha - \beta).\sigma(A)$ for every $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$;

(k) $\sigma(G_{\alpha,\beta})A)) \leq \sigma(A)$, for every $\alpha, \beta \in [0, 1]$;

(l) $\sigma(H_{\alpha,\beta}(A)) \leq \beta + (\alpha + \beta).\sigma(A)$, for every $\alpha, \beta \in [0, 1]$;

(m) $\sigma(H^{*}_{\alpha,\beta}(A)) \leq \beta + (1 - \beta).\sigma(A)$, for every $\alpha, \beta \in [0, 1]$;

(n) $\sigma(J_{\alpha,\beta}(A)) \leq \alpha + (\alpha + \beta).\sigma(A)$, for every $\alpha, \beta \in [0, 1]$;

(o) $\sigma(J^{*}_{\alpha,\beta}(A)) \leq \alpha + (1 - \alpha).\sigma(A)$, for every $\alpha, \beta \in [0, 1]$;

(p) $\sigma(!A) \geq 0$;

(q) $\sigma(?A) \geq 0$;

(r) $\sigma(P_{\alpha,\beta}(A)) \geq 0$, for every $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$;

(s) $\sigma(Q_{\alpha,\beta}(A)) \geq 0$, for every $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$.

The second norm, which we shall define for every propositional form A is:

$$\partial(A) = (\mu(A)^2 + \gamma(A)^2)^{1/2}$$

Thus defined the two norms are analogous to both basic classical types of norms.

For the norm ∂ the following assertions are valid.

THEOREM 2: For every two propositional forms A and B:

(a) $\partial(\neg A) = \partial(A)$;

(b) $\partial(A \& B) \geq \min(\partial(A), \partial(B))$;

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In the theory of fuzzy sets (see e.g. [10]) two different types of distances are defined, generated from the following metric

$$m(A, B) = |\mu(A) - \mu(B)|$$

for the propositional forms A and B.

Here the Hemming's and Euclid's metrics coincide. In the case of the intuitionistic fuzziness these metrics are different:

$$h(A, B) = \frac{1}{2} . (|\mu(A) - \mu(B)| + |\gamma(A) - \gamma(B)|)$$

(Heming's metric) and

$$e(A, B) = (\frac{1}{2} . ((\mu(A) - \mu(B))^{2} + (\gamma(A) - \gamma(B))^{2}))^{1/2}$$

(Euclid's metric).

When the equality

$$\gamma(A) = 1 - \mu(A),$$

is valid, both metrics are reduced to the metric m(A, B).

For proving, that h and e are pseudo-metrics over E in the

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For proving, that h and e are pseudo-metrics over E in the sense of [7], we must prove that for every three propositional forms A, B and C:

$$h(A, B) + h(B, C) \geq h(A, C),$$
$$h(A, B) = h(B, A),$$
$$e(A, B) + e(B, C) \geq e(A, C),$$
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The third equality, which characterizes the metrics (as above) is not valid. Therefore h and e are pseudo-mertrixes. The proofs of the above four equalities and inequalities are trivial.

*
* *

Using the geometrical interpretations of IFLs from [11, 12], we shall discuss other types of norms for a given proposition p (see Fig. 1 and 2).

For the case of the third interpretation we can calculate the area of the triangle as a function of the angles $\alpha(p)$ and $\beta(p)$.

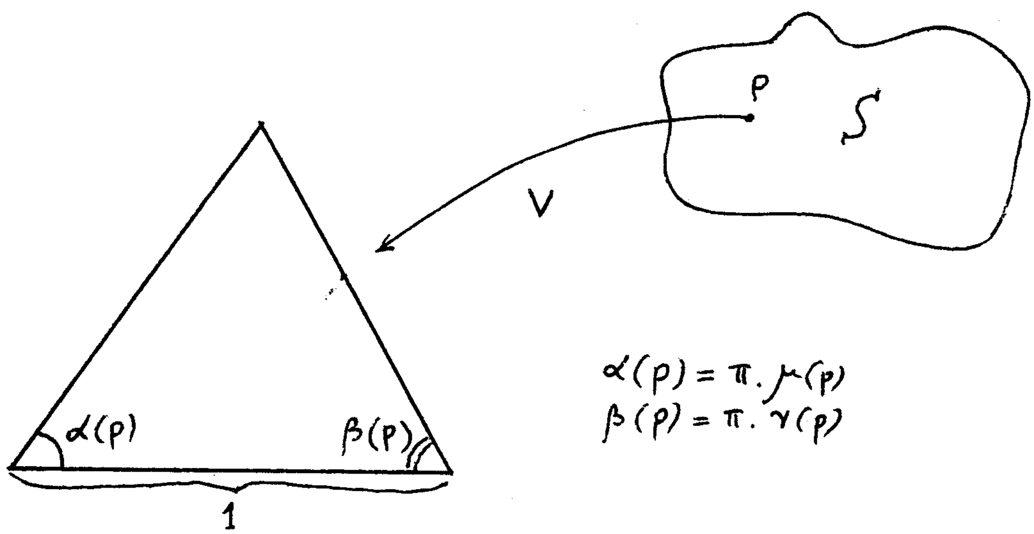
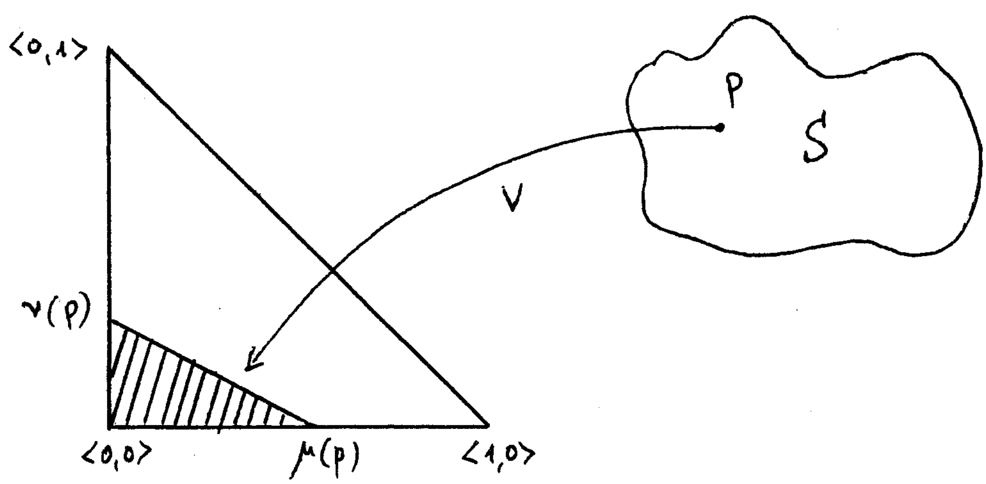


Fig. 1.



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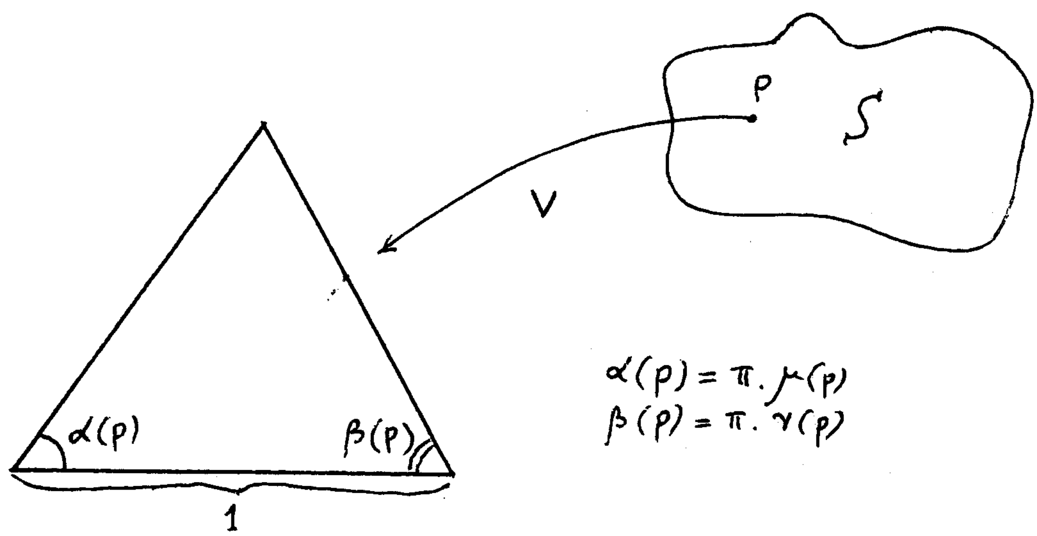

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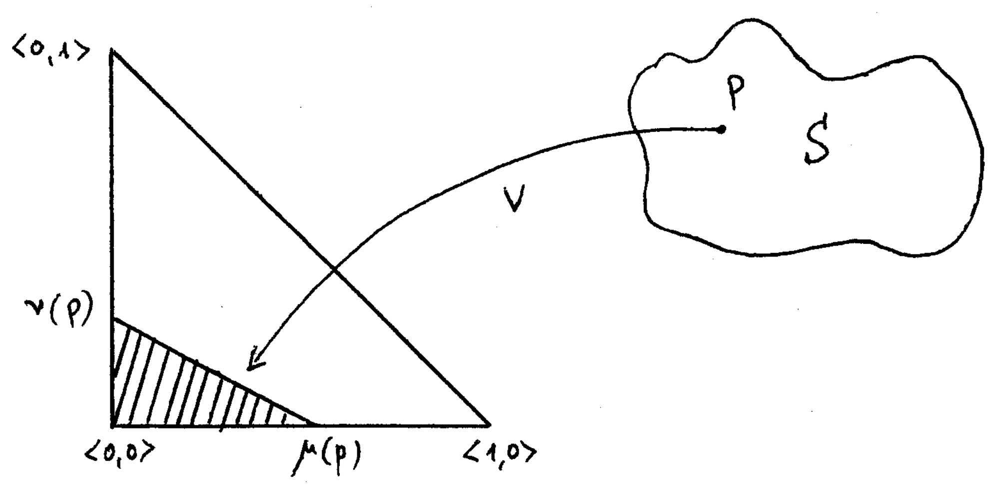

Fig. 2.

Then to the proposition p we can juxtapose the number

$$s_1(p) = (\cos(\alpha(p) - \beta(p)) - \cos(\alpha(p) + \beta(p)))/\sin(\alpha(p) + \beta(p)),$$

where $\alpha(p) = \pi.\mu(p)$ and $\beta(p) = \pi.\gamma(p)$ (here "π" is the mathematical constant "pi").

Obviously, in the case of the ordinary fuzzy sets, $\alpha(p) + \beta(p) = \pi$, i.e. $s(p) = \infty$.

For the case of the fourth interpretation we can calculate the area of the triangle as a function of the legs of the rectangular triangle with lenghts $\mu(p)$ and $\gamma(p)$. Then to the proposition p we can juxtapose the number

$$s_2(p) = \mu(p).\gamma(p).$$

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