QUASI-SYMMEDIAN VARIATIONAL INEQUALITIES FOR FUZZY MAPPINGS

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Abstract. In this paper some existence theorems of solutions of quasi- symmedian vaviational inequalities for fuzzy mappings are established. The obtained findings is a continuation of the shih-sen chang's papers [2,3].

Key words: quasi- symmedian variational inequality for fuzzy mapping.

1. Introduction and preliminaries

The purpose of this paper is to introduce the concept of quasi-symmedian Variational inequalities for fuzzy mapping and to obtain some existence theorems of Solutions of quasi -symmedian variational inequalities for fuzzy mappings. The obtained findings is a continuation of the shih-sen Chang's [2, 3].

Let M and N be two Hansdorff topological vector spaces and $X \subset M$, $Y \subset N$ be two nonempty closed convex subsets. Throughout this paper we always denote by $\mathfrak{F}(X)$ ($\mathfrak{F}(Y)$) the collection of all fuzzy sets on X(Y).
A mapping from X into $\mathfrak{F}(y)$($\mathfrak{F}(X)$) is called a fuzzy mapping. If $F: X \to (Y)$ is a fuzzy mapping, then for each $x \in X$, $F(x)$ (denote by F_x in the sequel) is a fuzzy set in $\mathfrak{F}(Y)$ and $F_x(y)$ is the degree of membership of point y in F_x. A fuzzy mapping $F: X \to \mathfrak{F}(Y)$ is called convex, if for each $x \in X$, the fuzzy set F_x on Y is a fuzzy convex set, i.e., for any $y_1, y_2 \in y$, $t \in [0,1]$

$$F_x(ty_1+(1-t)y_2) \geqslant \min(F_x(y_1), F_x(y_2)).$$

A fuzzy mapping $F: X \to \mathfrak{F}(Y)$ is called closed, if $F_x(y)$ is upper semi-continuous (as a function on $X \times Y$).

In the sequel, We denote by

$$(A)_\alpha = \{x \in X: A(x) \geqslant \alpha\}, \quad \alpha \in (0,1]$$

the α-cut set of $A \in \mathfrak{F}(X)$.

2. Quasi-symmedian Vasiational inequalitise for fuzzy mappings

Lemma 1. Let M and N be two Hansdorff topological vector spaces, X M, Y N be two nonempty compact convex subsets, and $\alpha : X \to (0,1]$ a lower semi-continuons function. Let $F: X \to \mathcal{F}(y)$ be a fuzzy mapping with $(F_x)_{\alpha(x)} \neq \Phi$ for each $x \in X$. Let $S: X \to z^y$ be a mapping defined by $S(x)=(F_x)_{\alpha(x)}$.

(i) If F is a convex fuzzy mapoing, then S is a mapping with nonempty convex Values;

(ii) If F is a closed convex fuzzy mapping, then s is an upper semi-continuous mapping with nonempty, close convex vlaues.

Proof. (i) By the assumptions, for each $x \in X$ $s(x) \neq \Phi$. Since F is a convex fuzzy mapping, for each $x \in X$ and for any $y, z \in S(x), t \in [0,1]$

$$F_x(ty+(1-t)z) \geqslant \min\{F_x(y), F_y(z)\} \geqslant \alpha(x).$$

This implies that $ty+(1-t)z \in (F_x)_{\alpha(x)} = s(x)$, i.e., $s(x)$ is convex.

(ii) For any $x \in X$ if $\{y_j\}_{j \in I}$ (I is an index set) is any net of $s(x)$ and $y_j \to y_0 \in Y$, thus $(x, y_j) \to (x, y_0) \in X \times Y$ and $F_x(y_j) \geqslant \alpha(x)$.

Since F is a closed fuzzy mapping, $F_x(y)$ is an upper semi-continuous function of (x, y). Hence we have

$$\alpha(x) \leqslant \overline{\lim_j} F_x(y_j) \leqslant F_x(y_0)$$

i.e., $y_0 \in s(x)$. This means that $s(x)$ is a closed set.

Since X and Y are compact sets and S is a closed Valued mapping, by a well-known result (cf. [1, pp 110-111]), the upper semi-continuity of S is equivalent to the close-ness of graph (s) (the graph of s). Therefore in order to prove the upper semi-continuity of S, it suffices to prove that the graph of S is closed.

Let $\{(x_j, y_j)\}_{j \in I}$ be any net of graph(s) and $x_j \to x_0 \in X$, $y_j \to y_0 \in Y$. Since F is closed,

$$\overline{\lim_j} F_x(y_j) \leqslant F_x(y_0). \qquad (2.1)$$

Besides, since $\alpha(x_j) \leqslant F_x(y_j)$ and α is lower semi-continuous, it follows from (2,1) that

$$\alpha(x_0) \leqslant F_x(y_0),$$

i.e. $(x_0, y_0) \in \mathrm{Graph}(s)$. This shows that graph(s) is a closed set of $X \times Y$.

Definition 1. Let N be a topological vector space. N is called quasi-complete, if for any bounded closed subset K of N is complete.

Remark. (i) It is easy to know that each Banach space is quasi-complete;

(ii) If N is a quasi-complete locally convex Hansdorff topological vector space and $K \subset N$ is a compact subset, then co(k) is also a compact subset of N(cf.[3,propositions 5.1.3]).

Theorem 1. Let M be a locally convex Hansdorff topological vector space and N a quasi- complete locally convex Hansdorff topological vector space. Let $X \subset M$ and $Y \subset N$ be two nonempty compact convex subsets. Let $F: X \to \mathcal{J}(Y)$ be a closed convex fuzzy mapping and $\alpha: X \to (0, 1]$ a lower semi -continuous function such that for each $x \in X$, $(F_x)_{\alpha(x)}$. is nonempty. Supose further that the function $\phi: X \times Y \times X \to R$ is continuous and satisfies the following conditions:

(i) $\phi(x,y,x) \geqslant 0$ for all $x \in X, y \in Y$;

(ii) for any given $(x,y) \in X \times Y$, $\phi(x,y,u)$ is quasi- convex in $u \in X$; Then there exist $x \in X$ and $y \in (F_x)_{\alpha(x)}$ such that quasi-symmedin variational inequalities

$$\phi(x,y,x) \geqslant 0 \text{ for all } x \in X.$$

Proof: First, we define a mapping $T: X \to z^y$ by

$$T(x) = (F_x)_{\alpha(x)}, x \in X.$$

By lemma 1, $T: X \to z^y$ is an upper semi- continuous mapping with nonempty compact convex valnes. Let

$$\Pi(x,y)\{s \in X: \phi(x,y,s) = \min_{u \in X} \phi(x,y,u)\}, (x,y) \in X \times Y.$$

since $\phi(x,y,u)$ is continuous and quasi-convex in u, $\Pi(x, y)$ is a closed convex subset of X. Since X is compact, $\Pi(x,y) \neq \Phi$ for all $(x,y) \in X \times Y$. This implies that $\Pi: X \times Y \to z^x$ is a mapping with nonempty compact convex values.

On the other hand, it is easy to show that $\Pi: X \times Y \to z^x$ is upper semi-continuous (this can be seen from [1, p. 111 ,corollary 9]).

Next , since X is compact and T is an upper semi -continuous mapping with nonempty compact convex values, by a well-known result (see [1,p.112,proposition 11]),we know that

$$T(x) = \bigcup_{x \in X} (F_x)_{\alpha(x)}$$

is a compact subset of Y. By Rewmark (ii) in the beginning of this section, $\overline{co}(T(X))$ is also a compact subset of Y.

Now we define a mapping P as follows:

$$P: X \times \overline{co}(T(X)) \to z^{x \times \overline{co}(T(X))}, P(x,y) = (\Pi(x,y), Tx).$$

therefore P is an upper semi-continuous mapping from a compact convex subset $X \times \overline{co}(T(X))$ into $z^{x \times \overline{co}(T(X))}$ with nonempty compact convex values. BY Fan-Glicksberg fixed point theorem, there exists a $(\bar{x},\bar{y}) \in X \times \overline{co}(T(X))$ such that $(\bar{x}, \bar{y}) \in P(\bar{x}, \bar{y})$. Hence we have $\bar{x} \in \Pi(\bar{x},\bar{y})$ and $\bar{y} \in Tx$. This implies that

$$y \in Tx, \phi(\bar{x},\bar{y},x) \geqslant \phi(\bar{x},\bar{y},\bar{x}) \geqslant 0 \text{ for all } x \in X.$$

This completes the proof.

Theorem 2. Let M, N, X, Y, F and α be the same as in theorem 1. Let $\xi: X\times Y\to M^*$(the dual of M) and $\eta: X\times Y \to M$ be two continuous mappings satisfying the following conditions:

(i) $\eta(x,x)=0$ for all $x\in X$;

(ii) for any given $(x,y)\in X\times Y$, $\langle \xi(x,y), \eta(u,x)\rangle$ is quasi-convex in $u\in X$.

Then there exist $\bar{x}\in X$ and $\bar{y}\in (F_{\bar{x}})_{\alpha(\bar{x})}$ such that symmedian variational inequalities

$$\langle \xi(\bar{x},\bar{y}), \eta(x,\bar{x})\rangle \geqslant 0 \quad \text{for all } x\in X.$$

Proof. Taking $\phi(x,y,x)=\langle \xi(x,y), \eta(u,x)\rangle$ in theorem 1, the conclusion follows from Theorem 1 immediately.

Theorem 3. Let M be a reflexive Banach space, N a quasi-complete locally convex Hansdorff topological vector space and $X\subset M$ and $Y\subset N$ be two nonempty close convex subsets. Let $F: X\to \mathcal{F}(y)$ be a convex fuzzy mapping and $F_x(y): X\times Y\to [0,1]$ as a function of (x,y) be upper semi- continuous in the weak topology of X and the topology of Y. Let $\alpha: X\to (0,1]$ be weakly lower semi-continuous. Suppose further that for each $x\in X$, $(F_x)_{\alpha(x)}$ is a nonempty compact subset of Y and that $\xi: X\times Y\to M^*$ is continuous from the weak topology of X and topology of Y to the norm topology of M^*. Suppose that $\eta: X\times Y\to M$ is a weakly continuous function satisfying the following conditions:

(i) $\eta(x,x)=0$ for all $x\in X$;

(ii) for any $(x,y)\in X\times Y$, $\langle \xi(x,y), \eta(u,x)\rangle$ is convex in $u\in X$;

(iii) there exists an $\bar{u}\in X$, $\|\bar{u}\|<r$ such that for any $x\in X$, $\|x\|=r$

$$\max_{y\in(F_x)_{\alpha(x)}} \langle \xi(x,y), \eta(\bar{u},x)\rangle \leqslant 0. \qquad (2.2)$$

Then there exist $x\in X$ and $\bar{y}\in (F_{\bar{x}})_{\alpha(\bar{x})}$ such that symmedian veriational inequalities

$$\langle \xi(x,y), \eta(x,x)\rangle \geqslant 0 \quad \text{for all } x\in X. \qquad (2.3)$$

Proof. Let the mapping $T: X\to z^y$ be definde by $Tx=(F_x)_{\alpha(x)}$. BY the assumptions and lemma 1, $T: X\to z^y$ is a mapping with nonempty compact convex valuesand it is upper semi-continuous from the weak topology of X to the topology of Y. Denote $X_r = X\cap B_r(0)$, where $B_r(0)=\{x\in M: \|x\|\leqslant r\}$, then X_r is a weakly compact convex subset of X. Letting

$$\phi(x,y,u)=\langle \xi(x,y), \eta(u,x)\rangle,$$

then $\phi: X_r\times Y\times X_r\to R$ is a continuous function in the weak topology of X_r to the topology of Y. By Theorem 1, there exist $x\in X_r$, $y\in (F_x)_{\alpha(x)}$ such that

$$\langle \xi(\bar{x},\bar{y}), \eta(x,\bar{x})\rangle \geqslant 0 \quad \text{for all } x\in X_r \qquad (2.4)$$

In the sequel, we shall discuss two cases:

(a). If $\|\bar{x}\|=r$, by condition (iii) and (2.4) we have

$$\langle \xi(\bar{x},\bar{y}), \eta(\bar{u},\bar{x})\rangle = 0 \qquad (2.5)$$

Hence for any given $x\in X$, taking $\lambda\in(0,1)$ which is little enough such that $w=\lambda x+(1-\lambda)\bar{u}\in X_r$, from (2.4) we have

$$0\leqslant\langle \xi(\bar{x},\bar{y}), \eta(w,\bar{x})\rangle \text{ (by condition (ii))}$$
$$\leqslant \lambda\langle \xi(\bar{x},\bar{y}), \eta(x,\bar{x})\rangle+(1-\lambda)\langle \xi(\bar{x},\bar{y}), \eta(\bar{u},\bar{x})\rangle$$
$$= \lambda\langle \xi(\bar{x},\bar{y}), \eta(x,\bar{x})\rangle.$$

(b). If $\|\bar{x}\|\leqslant r$, then for $x\in X$, taking $\lambda\in(0,1)$ which is little enough such that $Z=\lambda x+(1-\lambda)\bar{x}\in X_r$. By the same way as in (a), we can prove that

$$0\leqslant\lambda\langle \xi(\bar{x},\bar{y}), \eta(x,\bar{x})\rangle.$$

This completes the proof.

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