A note on g_λ—independent events

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Abstract: Borel—Cantelli's lemma with respect to g_λ—measures has been discussed in [5] and [11] when $\lambda<0$. In this paper, the lemma is obtained by different way when $\lambda\neq 0$ and some results with respect to g_λ—independent events, such as the analogues of the Borel zero—one criterion and the Kolmogorov zero—one law, ect., are established. In addition, we discuss the relationships between superadditive (subadditive) measures and belief (plausibility) functions, respectively.

keywords: g_λ—measure; subadditive (superadditive) measure; g_λ—independence; zero—one law.

1. Introduction

The concept of g_λ—meaasures, which drop the addity property and use the λ—additivity (see[6]) instead, was initiated by Sugeno [9]. The concept of g_λ—independence of two sets was introduced by Kruse [8]. The notion of similarity, g_λ—indenpendent class, as a generalization of above concept, was defined by Hua[5] and Zhang [11]and Borel—Cantelli's lemma with respect to g_λ—measures has been discussed by Hua [5] when $\lambda<0$. But a general lemma, $\lambda\neq 0$, is still lacking. Our main goal in this paper is to presente another proof of Borel—Cantelli's lemma and some zero—one law for g_λ—independent events when $\lambda\neq 0$.

In Setion 2, we discuss some elemental properties of g_λ—measures such as countably λ—subadditivity and the convergence or divergence of the series $\sum_{n=1}^{\infty} g_\lambda(A_n)$. In this section we also introduce the notion of subadditive and superadditive measure and study the relationships between superadditive (subadditive) measures and belief (plausibility) functions, respectively.

In Section 3, we introduce the notion of g_λ—indenpendent class and discuss its proerties. Furthermore, we obtain some results similar to classical probability theory, such as Borel — Cantelli's lemma, the Borel zero — one criterion and the Kolmogorov zero—one law, etc.

Throughout this paper, X denotes a nonempty set, $\mathscr{A}$ a σ—algebra on X and the pair $(X, \mathscr{A})$ a measurable space. We conventionalize that $\lambda \in (-1, +\infty)$.

2. Properties of the series composed of g_λ—measure—values

Definition 2.1 [9]. A set function g_λ from X to $[0,1]$ is called a g_λ—measure on $(X, \mathscr{A})$, if it satisfies the following conditions:

(1) $g_\lambda(\emptyset)=0$, $g_\lambda(X)=1$;

(2) $A, B \in \mathscr{A}, A \cap B = \emptyset \Rightarrow g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) g_\lambda(B)$;

(3) $\{A_n, n \geq 1\} \subset \mathscr{A}, A_n \uparrow A (A_n \downarrow A) \Rightarrow \lim_{n\to\infty} g_\lambda(A_n) = g_\lambda(A)$.

Obviously, a g_λ—measure on $(X, \mathscr{A})$ is monotone, that is . $g_\lambda(A) \leqslant g_\lambda(B)$ whenever $A \subset B, A, B \in \mathscr{A}$, and the equality $g_\lambda(A-B) = (g_\lambda(A) - g_\lambda(B)) / (1 + \lambda g_\lambda(B))$ holds for any sets $A, B \in \mathscr{A}$ and $B \subset A$. Moreover, g_λ—measures are probability measures when $\lambda = 0$. The properties of g_λ—measures on $(X, \mathscr{A})$ have been investigated in [3,4,6,11], etc. Here we recall some results which are useful in the following discussion.

Proposition 2.1[3], Let g_λ be g_λ—measure on $(X, \mathscr{A})$, then for every sequence $\{A_n, n \geq 1\}$ of disjoint sets in $\mathscr{A}$ we have

$$g_\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \begin{cases} \lambda^{-1}\left[\prod_{n=1}^{\infty}(1+\lambda g_\lambda(A_n)) - 1\right] & \lambda \neq 0, \text{(countably } \lambda\text{—additivity)} \\ \sum_{n=1}^{\infty} g_\lambda(A_n) & \lambda = 0. \end{cases}$$

Proposition 2.2[6]. If g_λ is a g_λ—measure on $(X, \mathscr{A})$, and $\lambda \neq 0$, then

$$g^* := \log_{1+\lambda}(1 + \lambda g_\lambda)$$

is a probability measure on $(X,\mathscr{A})$. Conversely, if g^* is a probability measure on $(X,\mathscr{A})$, then

$$g_\lambda := -\lambda^{-1} + \lambda^{-1}(1+\lambda)^{g^*}$$

is a g_λ—measure on $(X,\mathscr{A})$.

Now we introduce the notions of countably λ—subadditivity, subadditivity and superadditivity.

Proposition 2. 3. Let g_λ be a g_λ—measure on $(X,\mathscr{A})$, let $\lambda\neq 0$ and let $\{A_n, n\geqslant 1\}$ be arbitrary sequence of sets in $\mathscr{A}$, then for all $A\in\mathscr{A}$ with $A\subset\bigcup_{n=1}^{\infty}A_n$, we have

$$g_\lambda(A)\leqslant\lambda^{-1}\left[\prod_{n=1}^{\infty}(1+\lambda g_\lambda(A_n))-1\right] \text{ (countably } \lambda\text{—subadditivity).}$$

Proof. Let $A_0=\varnothing$ and $B_n=A_n-(\bigcup_{i=0}^{n-1}A_i)$, $n=1,2,\cdots$. Obviously, $B_n\subset A_n$, $n=1,2,\cdots$, $B_i\cap B_j=\varnothing$, $i\neq j$ and $\bigcup_{n=1}^{\infty}A_n=\bigcup_{n=1}^{\infty}B_n$, whenec by countably λ—additivily and monotonicity of g_λ, we get

$$g_\lambda(A)=g_\lambda(A\cap(\bigcup_{n=1}^{\infty}A_n))=\lambda^{-1}\left[\prod_{n=1}^{\infty}(1+\lambda g_\lambda(B_n\cap A))-1\right]$$
$$\leqslant\lambda^{-1}\left[\prod_{n=1}^{\infty}(1+\lambda g_\lambda(A_n))-1\right].$$

Definition 2. 2 . A nonnegative set function g on $(X,\mathscr{A})$ is called subadditive if for any sets $A,B\in\mathscr{A}$,

$$g(A\cup B)\leqslant g(A)+g(B)$$

A nonnegative set function g on $(X,\mathscr{A})$ is called superadditive if for any sets $A,B\in\mathscr{A}$, and $A\cap B=\varnothing$

$$g(A\cup B)\geqslant g(A)+g(B)$$

By this definition we can obtain the following propositions immediately.

Proposition 2. 4. If g is subadditive , then for arbitrary positive integer n, $\{A_i, 1\leqslant i\leqslant n\}\subset\mathscr{A}$,

$$g(\bigcup_{i=1}^{n}A_i)\leqslant\sum_{i=1}^{n}g(A_i).$$

If g is superadditive, then for arbitrary positive integer n, disjoint sets $\{A_i, 1\leqslant i\leqslant n\}$

$\subset \mathscr{A}$,

$$g(\bigcup_{i=1}^{n} A_i) \geqslant \sum_{i=1}^{n} g(A_i).$$

Proposition 2. 5. If g is subadditive and continuous from below ,i. e. for any increasing sequence $\{B_n, n \geqslant 1\}$ of sets in $\mathscr{A}$, $\lim_{n\to\infty} g(B_n) = g(\lim_{n\to\infty} B_n)$, then for any sequence $\{A_n, n \geqslant 1\}$ of sets in $\mathscr{A}$,

$$g(\bigcup_{n=1}^{\infty} A_n) \leqslant \sum_{n=1}^{\infty} g(A_n).$$

If g is superadditive and continuous from below, then for any sequence $\{A_n, n \geqslant 1\}$ of disjoint sets in $\mathscr{A}$,

$$g(\bigcup_{n=1}^{\infty} A_n) \geqslant \sum_{n=1}^{\infty} g(A_n).$$

Propsition 2. 6, If g_λ is a g_λ—measure on $(X, \mathscr{A})$, then g_λ is subadditive iff $\lambda \leqslant 0$ and it is superadditive iff $\lambda \geqslant 0$.

Now we recall two concepts and a result from [1]:

Definition 2. 3. A belief function on $(X, \mathscr{A})$ is a set function Bel $: \mathscr{A} \to [0,1]$ satisfying

(1) $Bel(\varnothing) = 0, Bel(X) = 1$

(2) $\forall\, n > 1$ *and* $\{A_i, 1 \leqslant i \leqslant n\} \subset \mathscr{A}$

$$\text{Bel}(\bigcup_{i=1}^{n} A_i) \geqslant \sum_{I \subset \{1,2,\cdots n\}, I = \varnothing} (-1)^{|I|-1} \text{Bel}(\bigcap_{i \in I} A_i).$$

A plausibility function on $(X, \mathscr{A})$ is a set function $\text{Pl}: \mathscr{A} \to [0,1]$ satisfying

(1) $\text{Pl}(\varnothing) = 0. \text{Pl}(X) = 1$,

(2) $\forall\, n > 1$ and $\{A_i, 1 \leqslant i \leqslant n\} \subset \mathscr{A}$

$$\text{Pl}(\bigcap_{i=1}^{n} A_i) \geqslant \sum_{I \subset \{1,2,\cdots n\}, I = \varnothing} (-1)^{|I|-1} \text{Pl}(\bigcup_{i \in I} A_i).$$

Proposition 2. 7. Suppose g_λ is a g_λ—measure on $(X, \mathscr{A})$, Then g_λ is a belief function iff $\lambda \geqslant 0$ and it is a plausibity function iff $\lambda \leqslant 0$.

Applying Proposition 2. 6 and Proposition 2. 7, we can obtain the following theorem immediately.

Theorem 2. 1. Suppose g_λ is a g_λ—measure on $(X, \mathscr{A})$. Then g_λ is a belief function iff it is superadditive and g_λ is a plausibility function iff it is subadditive.

In order to prove the main theorem in this section, first we give the following lemma.

Lemma 2. 1. For arbitrary real number $x\in[0,1]$, the following inequalities hold

(1) $\log_{1+\lambda}(1+\lambda x)\leqslant x$ and $\frac{\lambda^2}{\ln(1+\lambda)}x\leqslant\log_{1+\lambda}(1+\lambda x)$, $\quad -1<\lambda<0$

(2) $x\leqslant\log_{1+\lambda}(1+\lambda x)$ and $\log_{1+\lambda}(1+\lambda x)\leqslant\frac{\lambda^2 x}{\ln(1+\lambda)}$, $\quad \lambda>0$.

Proof. Let $f(x)=\log_{1+\lambda}(1+\lambda x)-x$, $x\in[0,1]$. We can verify that $f(x)$ is a convex function for every parameter $\lambda\in(-1,0)$. Thus for arbitrary $s,t\in[0,1]$ and $x,y\geqslant 0$, $x+y=1$, we have

$$f(xs+yt)\leqslant xf(s)+yf(t).$$

Now we choose $s=1, t=0$, then

$$f(x)\leqslant xf(0)+yf(1)=0.$$

This implies that the inequality $\log_{1+\lambda}(1+\lambda x)\leqslant x$ holds for all $x\in[0,1], \lambda\in(-1,0)$. We can also verify that $-f(x)$ is a convex function for every parameter $\lambda>0$. Analogously we can prove that the inequality $x\leqslant\log_{1+\lambda}(1+\lambda x)$ holds for all $x\in[0,1], \lambda>0$.

If $\lambda>0$, by the mean value theorem, we get

$$log_{1+\lambda}(1+\lambda x)=log_{1+\lambda}(1+\lambda x)-log_{1+\lambda}1=\frac{\lambda^2 x}{1+\theta\lambda x}\frac{1}{ln(1+\lambda)}\qquad(0<\theta<1)$$

Since $1+\theta\lambda x\geqslant 1$, this implies that the inequality

$$log_{1+\lambda}(1+\lambda x)\leqslant\frac{\lambda^2 x}{ln(1+\lambda)}.$$

holds for all $x\in[0,1], \lambda>0$. Analogously, the following inequality can be proved

$$\frac{\lambda^2 x}{ln(1+\lambda)}\leqslant log_{1+\lambda}(1+\lambda x),\qquad x\in[0,1],\qquad -1<\lambda<0.$$

and the lemma is proved.

From Lemma 2. 1 we can immediately present the main theorem in this section .

Theorem 2. 2. Suppose g_λ is a g_λ—measure on $(X,\mathscr{A})$ and g_λ^* as in Proposition 2. 2, then for every sequence $\{A_n, n\geqslant 1\}$ of sets in $\mathscr{A}$ and $\lambda\neq 0$

(1) $\sum_{n=1}^{\infty}g_\lambda(A_n)<\infty$ iff $\sum_{n=1}^{\infty}g_\lambda^*(A_n)<\infty$,

(2) $\sum_{n=1}^{\infty} g_\lambda(A_n)=\infty$ iff $\sum_{n=1}^{\infty} g_\lambda^*(A_n)=\infty$.

3. **g_λ—Independence events**

Throughout this section, we shall assume that $\lambda \neq 0$. We recall the following definition from[8].

Definition 3. 1. Let g_λ be a g_λ—measure on $(X, \mathscr{A})$. Sets A and B in $\mathscr{A}$ are called g_λ—independent iff

$$g_\lambda(A \cap B) = -\lambda^{-1} + \lambda^{-1}(1+\lambda)^{log_{1+\lambda}[1+\lambda g_\lambda(A)]log_{1+\lambda}[1+\lambda g_\lambda(B)]}$$

Obviously, A and B are g_λ—independent iff

$$log_{1+\lambda}[1 + \lambda g_\lambda(A \cap B)] = log_{1+\lambda}[1 + \lambda g_\lambda(A)]log_{1+\lambda}[1 + \lambda g_\lambda(B)]$$

Proposition 3. 1. If $A \in \mathscr{A}$ and $g_\lambda(A)=0$ or 1, then for arbitrary $B \in \mathscr{A}$, A and B are g_λ—independent.

Proof. If $g_\lambda(A)=0$, it follows from $0 \leqslant g_\lambda(A \cap B) \leqslant g_\lambda(A)$ that

$$g_\lambda(A \cap B) = 0 = -\lambda^{-1} + \lambda^{-1}(1+\lambda)^{log_{1-\lambda}[1+\lambda g_\lambda(A)]log_{1-\lambda}[1+\lambda g_\lambda(B)]}$$

If $g_\lambda(A)=1$, then $g_\lambda(A^c)=0$ and $0 \leqslant g_\lambda(A^cB) \leqslant g_\lambda(A^c)=0$, where A^c denotes the complement of A. Hence $g_\lambda(A^cB)=0$. Thus using Proposition 2. 2 we have

$$g_\lambda(A \cap B) = \frac{g_\lambda(B) - g_\lambda(A^cB)}{1 + g_\lambda(A^cB)} = g_\lambda(B)$$

$$= -\lambda^{-1} + \lambda^{-1}(1+\lambda)^{log_{1+\lambda}[1+\lambda g_\lambda(A)]log_{1-\lambda}[1+\lambda g_\lambda(B)]}$$

This completes the proof of Proposition 3. 1.

Proposition 3. 2. Set A is g_λ—independent of itself iff $g_\lambda(A)=0$ or 1.

Proof. By Proposition 3. 1 we can easily prove the proposition of sufficiency. If A and A are g_λ—indepentent, then

$$log_{1+\lambda}[1 + \lambda g_\lambda(A)] = (log_{1+\lambda}[1 + \lambda g_\lambda(A)])^2$$

that is. $g_\lambda^*(A)=(g_\lambda^*(A))^2$ or $g_\lambda^*(A)(1-g_\lambda^*(A))=0$. Hence $g_\lambda^*(A)=log_{1+\lambda}[1+\lambda g_\lambda(A)]=0$ or 1. And this implies $g_\lambda(A)=0$ or 1. This has proved the proposition of necessity.

Now we generalize the concept which two sets are g_λ—independent to that of g_λ

—independent classes.

Definition 3. 2. If g_λ is g_λ—measure on$(X,\mathscr{A})$ and T a non—empty index set, classes $\mathscr{L}_t$ of sets in $\mathscr{A}$, $t\in T$ are called g_λ—independent if for each integer $n\geqslant 2$, each choice of distinct $t_i\in T$, and sets $A_i\in\mathscr{L}_{t_i}$, $1\leqslant i\leqslant n$,

$$log_{1+\lambda}[1+\lambda g_\lambda(\bigcap_{i=1}^{n}A_i)]=\prod_{i=1}^{n}log_{1+\lambda}[1+\lambda g_\lambda(A_i)]$$

Sets $\{A_t,t\in T\}\subset\mathscr{A}$ are called g_λ—independent if the one—element classes $\mathscr{L}_t=\{A_t\}$, $t\in T$, are g_λ—independent.

Obviously, non—empty subclasses of g_λ—independent classes are likenwise g_λ—independent claeese. Conversely, if for every non—empty finite subset $T_1\subset T$, the classes $\mathscr{L}_t$, $t\in T$, are g_λ—independent, then so are the classes $\mathscr{L}_t$, $t\in T$.

$\mathscr{T}$h e o r e m 3. 1(Extension theorem of g_λ—independent classes). Suppose $\{\mathscr{L}_t,t\in T\}$ are g_λ—independent classes. If, for every $t\in T$, $\mathscr{L}_t$ is a π—class, i. e. $\mathscr{L}_t$ is closed under the formation of finite intersections, then $\{\sigma(\mathscr{L}_t),t\in T\}$ are also g_λ—independent classes, where $\sigma(\mathscr{L}_t)$ denotes the σ—algebra generated by $\mathscr{L}_t$.

Proof. See[5,12].

An immediate consequen of the above theorem is the following .

Corollary 3. 1 Sets $\{A_t,t\in T\}\subset\mathscr{A}$ are g_λ—independent iff the classes $\mathscr{L}_t=\{\varnothing, X, A_t, A_t^c\}$ are g_λ—independent.

Theorem 3. 2(Borel—Cantelli lemma). Let g_λ be a g_λ—measure on$(X,\mathscr{A})$and let $\{A_n,n\geqslant 1\}$ be a sequence of sets in $\mathscr{A}$.

(1) If $\sum_{n=1}^{\infty}g_\lambda(A_n)<\infty$, then $g_\lambda(\limsup_{n\to\infty}A_n)=0$.

(2)If $\{A_n,n\geqslant 1\}$ are g_λ—independent and $\sum_{n=1}^{\infty}g_\lambda(A_n)=\infty$, then $g_\lambda(\limsup_{n\to\infty}A_n)=1$.

Proof. (1). On the one hand, by Corollary 2. 1, we have

$$\prod_{n=1}^{\infty}[1+\lambda g_\lambda(A_n)]<\infty$$

On the other hand, if $\prod_{n=1}^{\infty}[1+\lambda g_\lambda(A_n)]=0$, then

$$\sum_{n=1}^{\infty} log_{1+\lambda}((1+\lambda g_\lambda(A_n)) = \sum_{n=1}^{\infty} g_\lambda^*(A_n) = \infty$$

This contradicts that g_λ^* is a probability measure on $(X, \mathscr{A})$. This contradiction indicates that $\prod_{n=1}^{\infty}((1+\lambda g_\lambda(A_n))\neq 0$. Therefore the infinite product $\prod_{n=1}^{\infty}((1+\lambda g_\lambda(A_n))$ is converge and $\lim_{k\to\infty}\prod_{n=k}^{\infty}((1+\lambda g_\lambda(A_n))=1$.

Hence, it follows from Proposition 2.3 that

$$g_\lambda(\lim_{n\to\infty} sup A_n) = \lim_{k\to\infty} g_\lambda(\bigcup_{n=k}^{\infty}(A_n)) \leqslant \lim_{k\to\infty} \lambda^{-1}[\prod_{n=k}^{\infty}(1+\lambda g_\lambda(A_n)) - 1]$$

And (1) holds.

(2) Obviously, for arbitrary integer $m>1$, $\{A_k \bigcap_{n=k+1}^{m} A_n^c, k=1,2,\cdots,m-1\}$ are disjoint sets in $\mathscr{A}$ and $\bigcup_{n=1}^{m} A_n \supset \bigcup_{k=1}^{m-1}(A_k \bigcap_{n=k+1}^{m} A_n^c)$. Using Proposition 2.2, Corollary 3.1 and Definition 3.2 for abritrary integer $m\geqslant 1$, we get

$$1 \geqslant log_{1+\lambda}[1+\lambda g_\lambda(\bigcup_{n=1}^{m} A_n)]$$

$$\geqslant log_{1+\lambda}\{1+\lambda g_\lambda[\bigcup_{k=1}^{m-1}(A_k \bigcap_{n=k+1}^{m} A_n^c)]\} = \sum_{k=1}^{m-1} log_{1+\lambda}[1+\lambda g_\lambda(A_k \bigcap_{n=k+1}^{m} A_n^c)]$$

$$= \sum_{k=1}^{m-1} log_{1+\lambda}[1+\lambda g_\lambda(A_k)] \cdot log_{1+\lambda}[1+\lambda g_\lambda(\bigcap_{n=k+1}^{m} A_n^c)]$$

$$\geqslant \sum_{k=1}^{m-1} log_{1+\lambda}[1+\lambda g_\lambda(A_k)] \cdot log_{1+\lambda}[1+\lambda g_\lambda(\bigcap_{n=k+1}^{\infty} A_n^c)]$$

implying

$$1 \geqslant \sum_{k=1}^{\infty} log_{1+\lambda}[1+\lambda g_\lambda(A_k)] \cdot log_{1+\lambda}[1+\lambda g_\lambda(\bigcap_{n=k+1}^{\infty} A_n^c)]$$

By Theorem 2.2, $\sum_{n=1}^{\infty} log_{1+\lambda}[1+\lambda g_\lambda(A_n)]=\infty$. Therefore divergence of the series requires

$$\lim_{k\to\infty} log_{1+\lambda}[1+\lambda g_\lambda(\bigcap_{n=k+1}^{\infty} A_n^c)] = 0$$

and so

$$\lim_{k\to\infty} g_\lambda(\bigcap_{n=k+1}^{\infty} A_n^c) = 0$$

Hence we have

$$g_\lambda(\lim_{n\to\infty} sup A_n) = \lim_{k\to\infty} g_\lambda(\bigcup_{n=k}^{\infty} A_n)$$

$$= \lim_{k\to\infty}[1 - g_\lambda(\bigcap_{n=k}^{\infty} A_n^c)]/[1 + \lambda g_\lambda(\bigcap_{n=k}^{\infty} A_n^c)] = 1$$

And the proof of theorem is completed.

As an immediate consequence of Theorem 3. 2, we have the following corollary

Corollary 3. 2 (Borel Zero—One Criterion). Let g_λ be a g_λ—measure on $(X, \mathscr{A})$ and let the sets $\{A_n, n \geq 1\} \subset \mathscr{A}$ be g_λ—independent. Then

(1) $g_\lambda(\limsup_{n\to\infty} A_n) = 0$ iff $\sum_{n=1}^{\infty} g_\lambda(A_n) < \infty$

(2) $g_\lambda(\limsup_{n\to\infty} A_n) = 1$ iff $\sum_{n=1}^{\infty} g_\lambda(A_n) = \infty$

Theorem 3. 3. (Kolmogorov Zero—One Law). Let g_λ be a g_λ—measure on $(X, \mathscr{A})$ and let the sets $\{A_n, n \geq 1\} \subset \mathscr{A}$ be g_λ—independent. Then for each $A \in \bigcap_{n=1}^{\infty} \sigma(A_n, A_{n+1}, \cdots)$, $g_\lambda(A) = 0$ or 1, where $\sigma(A_n, A_{n+1}, \cdots)$ denotes the σ—algebra generated by the sets $A_n, A_{n+1}, \cdots$.

Proof. Let $\mathscr{A}_n = \{$all finite intersections of sets in $\{A_n, A_{n+1} \cdots\}\}$, for arbitrary integer $n \geq 1$. It is easy to show that $\mathscr{A}_n$ are g_λ—independent classes. By Theorem 3. 1, $\{\sigma(A_1), \sigma(A_2), \cdots \sigma(A_n), \sigma(\mathscr{A}_n) = \sigma(A_n, A_{n+1}, \cdots)\}$ are also g_λ—independent classes. If $A \in \bigcap_{n=1}^{\infty} \sigma(A_n, A_{n+1}, \cdots)$. then $A \in \sigma(A_n, A_{n+1}, \cdots)$ and therefor the sets $\{A_1, A_2, \cdots A_{n-1}, A\}$ are g_λ—independent. From the remark below Definition 3. 2, the sets $\{A, A_1, A_2, \cdots\}$ are g_λ—independent. Using the same way as stated above we can prove that $\sigma(A), \sigma(A_1, A_2, \cdots)$, are g_λ—independent classes. But $A \in \sigma(A_1, A_2, \cdots)$, from $A \in \sigma(A)$ and $A \in \sigma(A_1, A_2, \cdots)$ it follows that A is g_λ—independent of itself. Hence, by Proposition 3. 2. $g_\lambda(A) = 0$ or 1, and we finishes the proof of theorem.

By Theorem 3. 3 and $\limsup_{n\to\infty} A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n \in \bigcap_{n=1}^{\infty} \sigma(A_1, A_2, \cdots)$, we immediately get the following

Theorem 3. 4 (Borel Zero—One Law). Let g_λ be a g_λ—measure on $(X, \mathscr{A})$ and let $\{A_n, n \geq 1\} \subset \mathscr{A}$ be g_λ—independent. Then $g_\lambda(\limsup_{n\to\infty} A_n) = 0$ or 1.

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