Convergence of fuzzy measures and fuzzy integrals on fuzzy sets

Deli Zhang
Dept. of Math. Jilin Prov. Inst. of Education,
Changchun, Jilin, 130022, P. R. China.
and
Caimei Guo
Dept. of Basic Science, Changchun Univ.
Changchun, Jilin, 130022, P. R. China.

Abstract: In this paper, we will have further investgation on fuzzy measures and fuzzy integrals on fuzzy sets. Some concepts such as "convergence", "uniform convergence", "convergence in integral", "weak convergence" are introduced for the sequence of fuzzy measures on fuzzy sets. Then the properties and relations of them are discussed. Furthermore, various kinds of convergence theorems of fuzzy integrals on fuzzy sets are extended.

Keywords: Fuzzy measure on fuzzy sets; Fuzzy integral on fuzzy sets; Generalized convergence theorem.

1. INTRODUCTION

Since Dr. Sugeno[4] brougt out the concepts of fuzzy measures and fuzzy integrals, the theory has been made deeper by Ralescu and Adams[3], Wang[5], and many others. But up to now, there is few discussions on the convergence of sequences of fuzzy measures. It is well known that there is a complete theory of weak convergence for classical measures. The purpose of this paper is try establishing the corresponding theory for fuzzy measures on fuzzy sets. Since a fuzzy measure is a monotone set-function, it has no additivity , of course, the problem is valuable and important.

Let X be a fixed set, P(X) is the power set of X, $\tilde{\mathscr{A}}$ is a fuzzy σ-algebra formed by the fuzzy subsets of X, and (X, $\tilde{\mathscr{A}}$) be a fuzzy measurable space. Let F(X) denote the set of all $\tilde{\mathscr{A}}$ -measurable functions from X to $[0, +\infty]$.

Definition 2. 1 A set-function $\mu: \tilde{\mathscr{A}} \to [0, +\infty]$ is said to be a fuzzy measure if it satisfies the following conditions:

(F_1): $\mu(\varphi)=0$;

(F_2): $A \subset B$ implies $\mu(A)\leqslant\mu(B)$;

(F_3): $An \uparrow A$ implies $\mu(An) \uparrow \mu(A)$;

(F_4): $An \downarrow A$, and there exists a n_o, s. t $\mu(An_o) < +\infty$ implies $\mu(An) \downarrow \mu(A)$.

The triplet $(X, \tilde{\mathscr{A}}, \mu)$ is called a fuzzy measure space. Let M(X) denote the set of all fuzzy mesures on $(X, \tilde{\mathscr{A}})$.

Definition 2. 2 Let $f\in F(X)$, $A\in\tilde{\mathscr{A}}$, $\mu\in M(X)$. Then the fuzzy integral of f on A with respect to μ is defined as

$$\int_A fd\mu = \bigvee_{\alpha\geqslant 0}[\alpha \wedge \mu(\chi_{F_\alpha} \cap A)].$$

where $F_\alpha=\{x\in X: f(x)\geqslant\alpha\}$.

Lemma 2. 1 Let μ_1, $\mu_2\in M(X)$, $A\in\tilde{\mathscr{A}}$. If we define $\mu_1 \vee \mu_2$, $\mu_1 \wedge \mu_2$ as

$$(\mu_1 \wedge \mu_2)(A) = \mu_1(A) \wedge \mu_2(A)$$

$$(\mu_1 \vee \mu_2)(A) = \mu_1(A) \vee \mu_2(A)$$

then $\mu_1 \vee \mu_2$, $\mu_1 \wedge \mu_2\in M(X)$.

Lemma 2. 2 (Transformation theorem of fuzzy integral) Let $f\in F(X)$, $A\in\tilde{\mathscr{A}}$, $\mu\in M(X)$. Then

$$\int_A fd\mu = \int_0^{+\infty} \mu(A \cap \chi_{F_\alpha})d\alpha$$

where α is the Lebesgue measure on $[0, +\infty]$, and the right integral $\int_0^{+\infty}\mu(A \cap \chi_{F_\alpha})d\alpha$ is also a fuzzy integral.

2. CONVERGENCE OF FUZZY MEASURES ON FUZZY SETS

In the section, the concepts of "convergence", "uniform convergence", "convergence in integral", and "weak convergence" are introduced. At the same time, their properties and relations are discussed.

Definition 2.1 Let $\{\mu_n\}$ be a sequence of fuzzy measures, i.e $\{\mu_n\}\subset M(X)$. We define the inferier limit and the superier limit, respectly as follow:

$$(\underline{lim}_{n\to\infty}\mu_n)(A) = \underline{lim}_{n\to\infty}\mu_n(A)$$

$$(\overline{lim}_{n\to\infty}\mu_n)(A) = \overline{lim}_{n\to\infty}\mu_n(A)$$

where $A\in\tilde{\mathscr{A}}$. If there exists a set-function $\mu:\tilde{\mathscr{A}}\to[0, +\infty]$, s.t. $\mu(A)=(\underline{\lim}_{n\to\infty}\mu_n)(A)=(\overline{\lim}_{n\to\infty}\mu_n)(A)$ for each $A\in\tilde{\mathscr{A}}$, then we say that $\{\mu_n\}$ is convergent to μ, simply written by $\lim_{n\to\infty}\mu_n=\mu$ or $\mu_n\to\mu$. If the convergence is uniform with $A\in\tilde{\mathscr{A}}$, then we say that $\{\mu_n\}$ is convergent uniformly to μ, which is simply written by $\mu_n \overset{U}{\to}\mu$.

Obviously, if $\mu_n\to\mu$, then μ is unique.

After the definition, we will naturally ask whether the set-function μ, i.e the limitation of $\{\mu_n\}$ is a fuzzy measure at the case of convergence or uniform convergence, then the following proposition will give the answer.

Proposition 2.1 Let $\{\mu_n\}\subset M(X)$, μ be a set-function from $\tilde{\mathscr{A}}$ to $[0, +\infty]$. Then

(i) $\mu_n\overset{U}{\to}\mu$ implies $\mu\in M(X)$

(ii) $\mu_n\to\mu$ doesn't imply $\mu\in M(X)$

Proposition 2.2 Let $\{\mu_n\}\subset M(X)$, $\mu\in M(X)$. If $\int_X f d\mu_n\to\int_X f d\mu$ holds for all $f\in F(X)$, then $\mu_n\to\mu$.

Proposition 2.3 Let $\{\mu_n\}\subset M(X)$, $\mu\in M(X)$, $f\in F(X)$, and $A\in\tilde{\mathscr{A}}$. If $\mu_n\uparrow\mu$, i.e. $\mu_1(A)\leqslant\mu_2(A)\leqslant\cdots$ for each $A\in\tilde{\mathscr{A}}$, and $\mu_n\to\mu$, then

$$\int_A f d\mu_n \uparrow \int_A f d\mu$$

Proposition 2.4 Let $\{\mu_n\}\subset M(X)$, $\mu\in M(X)$, $f\in F(X)$, and $A\in\tilde{\mathscr{A}}$. If $\mu_n\downarrow\mu$, i.e. $\mu_1(A)\geqslant\mu_2(A)\geqslant\cdots$ for each $A\in\tilde{\mathscr{A}}$, and $\mu_n\to\mu$, and there exists a n_0

s. t $\mu_{n0}(X) < +\infty$, then

$$\int_A f d\mu_n \downarrow \int_A f d\mu.$$

Proposition 2.5 Let $\{\mu_n\} \subset M(X)$, $\mu \in M(X)$, $f \in F(X)$, and $A \in \tilde{\mathscr{A}}$. If $\mu_n \to \mu$, and there exists a n_0, s. t $\mu_{n0}(X) < +\infty$, then

$$\int_A f d\mu_n \to \int_A f d\mu.$$

Corollary 2.1 Let $\{\mu_n\} \subset M(X)$, $\mu \in M(X)$. If there exists a n_0, s. t $\mu_{n0}(X) < +\infty$, then $\mu_n \to \mu$ iff the equation $\lim_{n\to\infty} \int_X f d\mu_n = \int_X f d\mu$ holds for all $f \in F(X)$.

Definition 2.2 Let $\{\mu_n\} \subset M(X)$, $\mu \in M(X)$. Then $\{\mu_n\}$ is said to be convergent to μ in integral if $\int_X f d\mu_n \to \int_X f d\mu$ holds for all $f \in F(X)$. It is simply written by $\mu_n \xrightarrow{I} \mu$.

Definition 2.3 Let $\{\mu_n\} \subset M(X)$, $\mu \in M(X)$. Then $\{\mu_n\}$ is said to be convergent weakly to μ if there exist a non-null $f \in F(X)$, s. t $\int_A f d\mu_n \to \int_A f d\mu$ hold for all $A \in \tilde{\mathscr{A}}$. It is simply written by $\mu_n \xrightarrow{W} \mu$.

From above discussions, we can easily obtain the following results, i. e the relations between various kinds of convergences.

Theorem 2.1 Let $\{\mu_n\} \subset M(X)$, $\mu \in M(X)$. Then

(i) $\mu_n \xrightarrow{U} \mu$ implies $\mu_n \to \mu$.

(ii) $\mu_n \xrightarrow{I} \mu$ implies $\mu_n \to \mu$.

(iii) $\mu_n \xrightarrow{I} \mu$ imlies $\mu_n \xrightarrow{W} \mu$.

Theorem 2.2 Let $\{\mu_n\} \subset M(X)$, $\mu \in M(X)$. If there exists a n_0, s. t $\mu_{n0}(X) < +\infty$, then

(i) $\mu_n \xrightarrow{U} \mu$ imlies $\mu_n \xrightarrow{I} \mu$.

(ii) $\mu_n \xrightarrow{I} \mu$ is equavolient to $\mu_n \to \mu$.

3. GENERALIZED CONVERGENCE THEOREM OF FUZZY INTEGRALS ON FUZZY SETS

In this section, our purpose is discussing the problem of the convergence of the sequence of fuzzy integral with respect to the sequence of fuzzy measures. The main results are the generilized Fatou' s lemmas and convergence theorems.

Theorem 3. 1 (Generalized monotone convergence theorem) Let $\{\mu_n(n\geqslant 1), \mu\}\subset M(X)$, $\{f_n(n\geqslant 1), f\}\subset F(X)$, $A\in\tilde{\mathscr{A}}$. If $f_n\uparrow f$, $\mu_n\uparrow\mu$, then

$$\int_A f_n d\mu_n \uparrow \int_A f d\mu.$$

Theorem 3. 2 (Generalized monotone convergence theorem)

Let $\{\mu_n(n\geqslant 1), \mu\}\subset M(X)$, $\{f_n(n\geqslant 1), f\}\subset F(X)$, $A\in\tilde{\mathscr{A}}$, and there exists a n_0, s. t $\mu_{n0}(X) < +\infty$. If $\mu_n\downarrow\mu$, $f_n\downarrow f$, then

$$\int_A f_n d\mu_n \downarrow \int_A f d\mu$$

Theorem 3. 3 (Generalized Fatou' s lemmas)

Let $\{\mu_n\}\subset M(X)$, $\{f_n\}\subset F(X)$, and $\{\bigwedge_{k=n}^{\infty}\mu_k\}\subset M(X)$, $\{\bigvee_{k=n}^{\infty}\mu_k\}\subset M(X)$, $A\in\tilde{\mathscr{A}}$

(i) If $\underline{\lim}_{n\to\infty}\mu_n\in M(X)$, then

$$\int_A (\underline{lim}_{n\to\infty} f_n) d(\underline{lim}_{n\to\infty}\mu_n) \leqslant \underline{lim}_{n\to\infty}\int_A f_n d\mu_n$$

(ii) If $\overline{\lim}_{n\to\infty}\mu_n\in F(X)$, and there exists a n_0, s. t $\mu_{n0}(X) < +\infty$, then

$$\overline{lim}_{n\to\infty}\int_A f_n d\mu_n \leqslant \int_A (\overline{lim}_{n\to\infty} f_n)\ d\ (\overline{lim}_{n\to\infty}\mu_n)$$

Theorem 3. 4 (Generilized Lebesgue convergence theorem)

Let $\{\mu_n(n\geqslant 1), \mu\}\subset M(X)$, $\{f_n(n\geqslant 1), f\}\subset F(X)$, and $\{\bigwedge_{k=n}^{\infty}\mu_k\}$, $\{\bigvee_{k=n}^{\infty}\mu_k\}\subset$

M(X). Further, assume there exists a n_0, s. t $\mu_{n0}(X) < +\infty$. If $f_n \to f$, $\mu_n \to \mu$, then

$$\int_A f_n d\mu \to \int_A f d\mu.$$

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