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THE POWERS AND COUNTABLE FUZZY CARDINALS OF FUZZY SETS

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ABSTRACT

In this paper, the equinumerous relation and the powers of fuzzy sets are defined, and some propositions about them are discussed. Then the countable fuzzy cardinals (ordinals) is defined on the natural number set N. The method to find the fuzzy cardinals of a fuzzy set defined on a countable universe of discourse X and the properties of this kind of cardinals are offered at last. All of these definitions coincide the relative definitions in classical set theory respectively.

Keywords: power of fuzzy set, countable fuzzy cardinals, countable fuzzy cardinals of fuzzy set.

Power and cardinals are the essential concepts in classical set theory. D. Dubois[1] offered the definition of scalar cardinals of a fuzzy set on finite universe of discourse. L. A. Zadeh defined the fuzzy cardinals by a fuzzy number (see [1]), H. X. Li and other authors[2,3] gave two kinds of fuzzy cardinals by using the equinumerous relations of λ-cuts and strong λ-cuts separately. In this paper, new definitions of power and cardinals of fuzzy set are given. They are compared with the former definitions.

Definition 1 Suppose $A \in \mathscr{F}(X)$, $A(x) = \lambda$. The fuzzy point x^{λ} on X is called the fuzzy shell point of A, denoted $x^{\lambda} \in A$. $A_{\bar{\lambda}} = \{x \mid x^{\lambda} \in A\} = \{x \mid A(x) = \lambda, x \in X\} \in \mathscr{P}(X)$ is called the λ-shell point set.

Obviously, $A = \{x^{\lambda} \mid x \in X\}$, $\mathrm{Ker}A = A_1$, $A_0 = \mathrm{Supp}A = \bigcup_{\lambda \in (0,1]} A_{\bar{\lambda}}$.

Definition 2 Let A, B be fuzzy sets. A and B are called equinumerous fuzzy sets if $\forall \lambda \in (0, 1]$, there exists a bijection $f_{\bar{\lambda}}: A_{\bar{\lambda}} \to B_{\bar{\lambda}}$, denoted $\overline{A} = \overline{B}$. B is prior to A if $\forall \lambda \in (0, 1]$ there exists a injection $f_{\bar{\lambda}}: A_{\bar{\lambda}} \to B_{\bar{\lambda}}$, denoted $\overline{A} \leqslant \overline{B}$. The equivalent class of fuzzy sets family, divided by the equivalent relation $=$, is called the power of fuzzy set A if A belongs to this equivalent class. We denoted the power of A by $\overline{A}$.

Theorem 1 (The Cantor-Bernstein Theorem for Fuzzy Sets) A and B are fuzzy sets. $\overline{A} = \overline{B}$ iff $\overline{A} \leqslant \overline{B}$ and $\overline{B} \leqslant \overline{A}$.

Proof Use the Cantor-Bernstain Theorem for $A_{\bar{\lambda}}$ and $B_{\bar{\lambda}}$ (omited).

By Theorem 1, relation $\leqslant$ is anti-symmetric. Obviously $\leqslant$ is transitive and reflexive, then $\leqslant$ is partial ordering.

Theorem 2 A and B are fuzzy sets. $\overline{A} = \overline{B}$ iff there exists a bijection $g: A_0 \to B_0$, and g is satisfied $(\forall x \in A_0)(B(g(x)) = A(x))$.

Proof Let $g = \bigcup_{\lambda \in (0,1]} f_{\bar{\lambda}}$ and $f_{\bar{\lambda}} = g|_{A_{\bar{\lambda}}}$ to prove (omited).

Lemma 1 A and B are fuzzy sets, Then $\overline{A} \leqslant \overline{B}$ iff there exists an injection $g: A_0 \to B_0$, and g is satisfied $(\forall x\in A_0)(B(g(x))=A(x))$.

Lemma 2 $A, B\in\mathscr{F}(X)$, and $A_{\bar{0}}=B_{\bar{0}}$. Then $\overline{A}=\overline{B}$ iff there exists a transformation g of X which is satisfied $(\forall x\in X)(B(g(x))=A(x))$.

Lemma 3 $A\in\mathscr{F}(X)$, g is a transformation of A_0, and $g(A)\in\mathscr{F}(X)$:

$$(g(A))(x)=\begin{cases}A(g^{-1}(x)), & x\in A_0;\\ 0, & \text{otherwise},\end{cases}$$

then $\overline{A}=\overline{g(A)}$.

Lemma 4 Let A and B be fuzzy sets, A_0 and B_0 be finite, then $\overline{A}=\overline{B}$ implies $|A_0|=|B_0|=n$ and $\sum_{i=1}^{n}A(x_i)=\sum_{j=1}^{n}B(y_j)$. The converse is not true.

Remark 1 In Reference [1] the power, or cardinals, of a fuzzy set A is defined as $|A|=\sum_{i=1}^{n}A(x_i)$. According to Lemma 4, it shows that the condition of this definition is weaker than that of Definition 2. But it is difficult to extend the former into the case of fuzzy set on infinite support set and to keep the extention coinciding with the classical definitions.

Theorem 3 Let A, B be fuzzy sets, If $\overline{A}=\overline{B}$ then

(1) $\forall\lambda\in(0,1]$, there exists a bijection $f_\lambda: A_\lambda\to B_\lambda$;

(2) $\forall\lambda\in(0,1]$, there exists a bijection $f_\lambda: A_{\underline{\lambda}}\to B_{\underline{\lambda}}$.

The converse is not true.

Proof To Prove using $A_\lambda=\bigcup_{\xi\geq\lambda}A_{\bar{\xi}}$ and $A_{\underline{\lambda}}=\bigcup_{\xi>\lambda}A_{\bar{\xi}}$ respectively (omited). The following example shows that the converse is not true. Let $X=[0,2]$, $A=X$, $B\in\mathscr{F}(X)$, $B(x)=1$ when $x\in[0,1]$ and $B(x)=0.5$ when $x\in(1,2]$. A, B are satisfied (1) and (2), but $\overline{A}\neq\overline{B}$.

Remark 2 The above example shows that the condition of Definition 2 is more stritic than those in Reference [2] and [3], but the former could not result the equinumerous of a classical set and a proper fuzzy set.

Theorem 4 Let A, B be fuzzy sets, then $\overline{A}=\overline{B}$ implies Ran$A$=Ran$B$, The converse is not true. Where Ran$A=\{\lambda\mid A(x)=\lambda, x\in X\}\in\mathscr{P}([0,1])$.

Lemma 5 Let A, B be fuzzy sets. If (1) Ran$A\neq$Ran$B$, or (2) Hgt$A\neq$Hgt$B$, or (3) Dph$A\neq$Dph$B$ then $\overline{A}\neq\overline{B}$, Where Hgt$A=\mathrm{Sup}_{x\in X}A(x)$ and Dph$A=\mathrm{Inf}_{x\in X}A(x)$.

Lemma 6 Let $A\in\mathscr{F}(X)$, $B\in\mathscr{F}(Y)$ and Ran$A$=Ran$B$.

(1) If $A(x)$ and $B(x)$ are strict monotone membership functions then $\overline{A}=\overline{B}$.

(2) If A and B are strict convex fuzzy sets, the branches of their membership functions are continuous and strict monotone then $\overline{A}=\overline{B}$.

(3) If the branches of the membership functions of A and B are continuous, stritic monotone and the ranges of corresponding branches are the same respectively, then $\overline{A}=\overline{B}$.

Definition 3 $N=\{0,1,2,\cdots\}$. $A\in\mathscr{F}(N)$ is called a countable fuzzy cardinals if the membership function $A(n)$ is decreasing over N. Fuzzy cardinals A is called finite if Supp$A$ is finite.

Suppose the universe of discourse X is countable or finite, $A\in\mathscr{F}(X)$, the fuzzy shell points of $A=\{x^\lambda\mid x\in X\}$ are countable or finite, then we could ordering them according to the value of λ as following:

$$x_0^{\lambda_0}, x_1^{\lambda_1}, \cdots, x_i^{\lambda_i}, \cdots.$$

Where $1\geq\lambda_0\geq\lambda_1\geq\cdots\geq\lambda_i\geq\cdots>0$. Obviously, $x_i\in A_0$. The mapping

$$f_A: A_0 \to N, f_A(x_0)=0, f_A(x_1)=1, \cdots, f_A(x_i)=i, \cdots$$

satisfies one of the two properties as following:

(1) If $A_0=\{x_0,x_1,\cdots,x_{m-1}\}$, then f_A is a bijection from A_0 to $f(A_0)=\{0,1,2,\cdots,m-1\}=M$.

(2) If A_0 is (infinite) countable, then f_A is a bijection from A_0 to $f(A_0)=N$.

The mapping f_A is really the mapping to permute the points of A_0 onto N according to the value of λ from big to small. We call f_A the decreasing permutal mapping from A to N (DP mapping). We also denote the mapping f_A by f in short.

Definition 4 Let X be a countable or finite universe of discourse. $A \in \mathscr{F}(X)$. f is a DP mapping from A to N. Card$A \in \mathscr{F}(N)$ is called the fuzzy cardinals of A, $\forall n \in N$,

$$(\text{Card}A)(n)=\begin{cases} A(f^{-1}(n)), & n \in f(A_0); \\ 0, & \text{otherwise.} \end{cases}$$

According to the rule of DP mapping f and Definition 4, Card$A$ is a fuzzy cardinals. By using Theorem 2, it is easy to prove that $\overline{A}=\overline{\text{Card}A}$. At last, Card$A$ is the unique fuzzy cardinals determined by A. Thus Definition 4 is reasonable.

Property 1 (1) If $A \in \mathscr{P}(X)$ then Card$A$ is a classical cardinals (ordinals).

(2) $\forall \lambda \in (0,1]$, $(\text{Card}A)_\lambda=\overline{A_\lambda}$, i. e., the natural number (ordinals) presented by $(\text{Card}A)_\lambda$ is just the number of elements in A_λ.

(3) $\overline{A}=\overline{B}$ iff Card$A$=Card$B$.

(4) $A, B \in \mathscr{F}(X)$, $A \subseteq B$ implies Card$A \subseteq$ Card$B$.

The proof is omited.

Remark 3 The following definition offered by L. A. Zadeh (introduced in Reference [1]):

$$|A|_f=\Sigma\alpha/|A_\alpha|=\{(n,\alpha), n \in N, \alpha=\text{Sup}\{\lambda, |A_\lambda|=n\}.$$

Where Supp$A$ is finite, Card$A$ and $|A|_f$ are both the fuzzy sets on N. But Definition 4 keeps up and extends an essential demand in classical set theory, i. e., the cardinals of a set is the ordinals (set) which is equinumerous to the set. There exists the difference between the two kinds of cardinals in formal. For example, $A=1/x_1+0.7/x_2+0.7/x_3$, then

$$|A|_f=1/1+0.7/3,$$
$$\text{Card}A=1/0+0.7/1+0.7/2,$$
$$(\text{Card}A)_{\overline{0.7}}=\{0,1,2\}=3=\overline{A_{0.7}}.$$

Other problems such as the fuzzy cadinals calculus will be discussed in another paper.

References

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