

M-Fuzzy Sets and Category of M-Fuzzy Sets

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INTRODUCTION

The theory of fuzzy set and the topoi theory have intimate connections. This has been shown to be true by various investigators[1-5]. It is well known that the concept of topos can lead to the logic operations of the class sets and there is power object in a topos. However, category Set^H of fuzzy sets given by Goguen is not a topos because it has no SC(Subobject Classifier). In earlier paper[1-2], we have shown that although category Set^H is not a topos, subcategory Fuz of Set^H can form a WTopos by the use of two new objects--Middle object and WSC . By the use of category Fuz and WTopos , the logic operators of fuzzy sets as defined by Zadeh can naturally be obtained and Fuz has powers. In this way, WTopos has similiar functions to topos.

As a example of WTopos , we introduce concept of M-fuzzy set and a category M-Fuz of M-fuzzy set. The topoi properties of category M-Fuz are investigated. Although M-Fuz is not a topos, it has two special objects--middle object and WSC . By the use of middle object and WSC , we can show that M-Fuz also forms a WTopos and M-Fuz has power objects.

1. Topos and WTopos

A topos is a category \mathcal{E} satisfying the following conditions:

- (1) Equalizers, finite products and exponentials exist in \mathcal{E} .
- (2) Terminal object U exists in \mathcal{E} , i.e. for each object A , there is one and only one morphism from A to U .
- (3) There is a SC in \mathcal{E} , i.e., there is an object Ω and a morphism ν from U to Ω such that for each monomorphism m from A' to A there is exactly one morphism ϕ_m from U to Ω so that the figure 1 is a pullback.

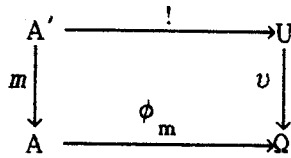


Figure 1

For example, category Set of class sets is a topos, $U=1=\{0\}$ is a terminal object; $\Omega=2=\{0,1\}$, $v:U \longrightarrow \Omega$, $v(0)=1$ is a SC. Then for any monomorphism $m:A' \longrightarrow A$, $\phi_m(a) = \begin{cases} 1, & \text{if } a \in m(A') \\ 0, & \text{else} \end{cases}$ is the characteristic function of A' .

Category Fuz of fuzzy sets is defined by:

object is (A, α) , where A is any set and α any mapping from A to $(0, 1]$; a morphism f from (A, α) to (B, β) is a mapping $f:A \longrightarrow B$ satisfying $\alpha(a) \leq \beta(f(a)), \forall a \in A$.

Fuz has the topoi properties such as

- (i) Equalizers, finite product and exponential exist in Fuz.
- (ii) Terminal object exists in Fuz.
- (iii) Fuz has no SC.

So Fuz is not a topos. However, category Fuz has two special objects:

(ii)' Middle object exists in Fuz, i.e., there is an object (I, α_I) such that for any object (A, α) in Fuz, there is exactly one morphism $f:A \longrightarrow I$ satisfying $\alpha_I(f(a)) = \alpha(a), \forall a \in A$. (Where $I = (0, 1], \alpha_I(\lambda) = \lambda$).

(iii)' There is an object (J, α_J) and a morphism $1_I:I \longrightarrow J$ such that for any monomorphism $m:(A', \alpha') \longrightarrow (A, \alpha)$ there is exactly one morphism α_m from (A, α) to (J, α_J) satisfying (1) $\alpha_m(a) \leq \alpha(a), \forall a \in A$;

(2) The figure 2 is a pullback.

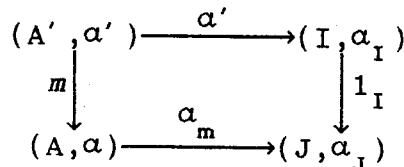


Figure 2

where $J=[0, 1], \alpha_J(\lambda)=1, 1_I(\lambda)=\lambda, \alpha_m(a) = \begin{cases} \alpha'(a'), & \text{if } a=m(a') \in m(A') \\ 0, & \text{else} \end{cases}$

the α_m can be seen as the membership function of fuzzy set

(A', α') . So α_m is called as MF of $\{(A', \alpha'), m\}$.

Category Fuz satisfying (i), (ii)' and (iii)' is called as a WTopos.

2. Category M-Fuz is also a WTopos

Let $M=(M, *, e)$ be a monoid and A be any set. (A, λ) is called as a M-set if (1) $\lambda: M \times A \longrightarrow A$ is a mapping, (2) $\lambda(e, a) = a$, (3) $\lambda(m, \lambda(n, a)) = \lambda(m * n, a), \forall a \in A, \forall m, n \in M$. Category M-Set of M-Sets is a topos.

Let (A, α) be an object of category Fuz, (A, α, λ) is called as a M-fuzzy set if (i) (A, λ) be a M-set, (ii) $\alpha(\lambda(m, a)) = \alpha(a), \forall a \in A, m \in M$.

Let M-Fuz be a category. Its object is M-fuzzy set (A, α, λ) ; a morphism from (A, α, λ) to (B, β, μ) be a mapping $f: A \longrightarrow B$ satisfying (1) $\mu(m, f(a)) = f(\lambda(m, a))$, (2) $\beta(\mu(m, f(a))) \geq \alpha(\lambda(m, a)), \forall a \in A, m \in M$.

Theorem 1. Category M-Fuz has all topos properties except one for it has no SC.

Let $\lambda_I(m, h) = h, \forall h \in I$, then for any M-fuzzy set (A, α, λ) there is exactly one morphism f from (A, α, λ) to (I, α_I, λ_I) such that $f(\lambda(m, a)) = \alpha(\lambda(m, a))$. (I, α_I, λ_I) be a middle object of category M-Fuz.

Let $\lambda_J(m, h) = h, \forall m \in M, h \in J$. Then we have

Theorem 2. For any monomorphism $\phi: (A', \alpha', \lambda') \longrightarrow (A, \alpha, \lambda)$ there is exactly one morphism α_ϕ satisfying

(i) $\alpha_\phi(\lambda(m, a)) \leq \alpha(\lambda(m, a))$;

(ii) The figure 3 is a pullback.

$$\begin{array}{ccc}
 (A', \alpha', \lambda') & \xrightarrow{\alpha'} & (I, \alpha_I, \lambda_I) \\
 \phi \downarrow & & \downarrow \lambda_I \\
 (A, \alpha, \lambda) & \xrightarrow{\alpha_\phi} & (J, \alpha_J, \lambda_J)
 \end{array}$$

Figure 3

Where $\alpha_\phi(\lambda(m, a)) = \begin{cases} \alpha'(\lambda(m, a')), & \text{if } a = \phi(a') \in \phi(A') \\ 0 & , \text{ else} \end{cases}$

α_ϕ is called as MMF of $\{(A', \alpha', \lambda'), \phi\}$.

We know from theorem 2 and definition of WTopos that category M-Fuz is also a WTopos.

3. Category M-Fuz Has Power Objects

A category Ξ with finite product is said to have power objects if to each Ξ -object A there Ξ -object $\mathbb{P}(A)$, \mathbb{E}_A and a monomorphism $\psi: \mathbb{E}_A \longrightarrow \mathbb{P}(A) \times A$, such that for any Ξ -object B and "relation" $r: R \longrightarrow B \times A$, there is exactly one Ξ -morphism $f_r: B \longrightarrow \mathbb{P}(A)$ for which there is a pullback in Ξ of the figure 4.

$$\begin{array}{ccc} R & \xrightarrow{r} & B \times A \\ \downarrow & & \downarrow f_r \times 1_A \\ \mathbb{E}_A & \xrightarrow{\psi} & \mathbb{P}(A) \times A \end{array}$$

Figure 4

Let (A, α, λ) and (B, β, μ) be two M-fuzzy sets, a "relation" from (A, α, λ) to (B, β, μ) is a M-fuzzy set (R, α_R, λ_R) satisfying $\lambda_R(m, (b, a)) = (\mu(m, b), \lambda(m, a))$ and $\alpha_R(b, \lambda(m, a)) = \alpha_R(\mu(m, b), a) \leq \text{Min}\{\alpha(a), \beta(b)\}$. Then we have

Theorem 3. In category M-Fuz, for any object (A, α, λ) , there are object $(\mathbb{F}_A, \alpha_F, \lambda_F)$, $(\mathbb{E}_A, \alpha_E, \lambda_E)$ and monomorphism $\text{id}: (\mathbb{E}_A, \alpha_E, \lambda_E) \longrightarrow (\mathbb{F}_A, \alpha_F, \lambda_F) \times (A, \alpha, \lambda)$, such that for any object (B, β, μ) and "relation" (R, α_R, λ_R) , there exactly one morphism $f_r: (B, \beta, \mu) \longrightarrow (\mathbb{F}_A, \alpha_F, \lambda_F)$ satisfying (1). $f(\mu(n, b))(\lambda(m, a)) \leq \alpha_R(\mu(n, b), \lambda(m, a))$; (2). The figure 5 is a pullback.

$$\begin{array}{ccc} (R, \alpha_R, \lambda_R) & \xrightarrow{r} & (B, \beta, \mu) \times (A, \alpha, \lambda) \\ \downarrow & & \downarrow f_r \times 1_A \\ (\mathbb{E}_A, \alpha_E, \lambda_E) & \xrightarrow{\text{id}} & (\mathbb{F}_A, \alpha_F, \lambda_F) \times (A, \alpha, \lambda) \end{array}$$

Figure 5

Where $\mathbb{F}_A = \{f \mid f: A \longrightarrow [0, 1] \text{ is a mapping satisfying } (*)\}$

$$(1) f(a) \leq \alpha(a), (2) f(\lambda(m, a)) \leq f(a), \forall a \in A, m \in M \quad (*)$$

$$\lambda_F(m, f)(a) = f(\lambda(m, a)), \forall m \in M, a \in A, f \in \mathbb{F}_A; \alpha_F(f) = 1, \forall f \in \mathbb{F}_A$$

$$\mathbb{E}_A = \{ \langle f, a \rangle \mid f \in \mathbb{F}_A, a \in A \text{ and } f(a) \neq 0 \}$$

$$\lambda_E(m, \langle f, a \rangle) = \langle \lambda_F(m, f), a \rangle, \alpha_E(\langle f, a \rangle) = f(a)$$

$$r(b, a) = (b, a), \forall (b, a) \in R, \text{id}(\langle f, a \rangle) = \langle f, a \rangle, \forall \langle f, a \rangle \in E_A$$

$$f_r(b)(a) = \begin{cases} \alpha_R(b, a), & \text{if } (b, a) \in R \\ 0 & , \text{if } (b, a) \notin R \end{cases}$$

We know from theorem 3 and definition of power object that category M-Fuz has power object.

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