M-Fuzzy Sets and Category of M-Fuzzy Sets

Xue-hai YUAN
Dept. of Mathematics, Liaoning Normal University,
Dalian 116022, *P.R.China*

INTRODUCTION

The theory of fuzzy set and the topoi theory have intimate connections. This has been shown to be true by various investigators[1-5]. It is well known that the concept of topos can lead to the logic operations of the class sets and there is power object in a topos. However, category SetH of fuzzy sets given by Goguen is not a topos because it has no SC(Subobject Classifier). In earlier paper[1-2], we have shown that although category SetH is not a topos, subcategory Fuz of SetH can form a WTopos by the use of two new objects--Middle object and WSC. By the use of category Fuz and WTopos, the logic operators of fuzzy sets as defined by Zadeh can naturally be obtained and Fuz has powers. In this way, WTopos has similiar functions to topos.

As a example of WTopos, we introduce concept of M-fuzzy set and a category M-Fuz of M-fuzzy set. The topoi properties of category M-Fuz are investigated. Although M-Fuz is not a topos, it has two special objects--middle object and WSC. By the use of middle object and WSC, we can show that M-Fuz also forms a WTopos and M-Fuz has power objects.

1. Topos and WTopos

A topos is a category Ξ satisfying the following conditions:

(1) Equalizers, finite products and exponentials exist in Ξ.

(2) Terminal object U exists in Ξ, i.e. for each object A, there is one and only one morphism from A to U.

(3) There is a SC in Ξ, i.e., there is an object Ω and a morphism υ from U to Ω such that for each monomorphism m from A' to A there is exactly one morphism ϕ_m from U to Ω so that the figure 1 is a pullback.

$$\begin{array}{ccc} A' & \xrightarrow{\ !\ } & U \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \upsilon} \\ A & \xrightarrow{\ \phi_m\ } & \Omega \end{array}$$

Figure 1

For example, category Set of class sets is a topos, $U=1=\{0\}$ is a terminal object; $\Omega=2=\{0,1\}$, $\upsilon: U \longrightarrow \Omega$, $\upsilon(0)=1$ is a SC. Then for any monomorphism $m: A' \longrightarrow A$, $\phi_m(a)=\begin{cases}1, \text{if } a\in m(A') \\ 0, \text{else}\end{cases}$ is the characteristic function of A'.

Category Fuz of fuzzy sets is definied by:

object is (A,α), where A is any set and α any mapping from A to $(0, 1]$; a morphism f from (A,α) to (B,β) is a mapping $f: A \longrightarrow B$ satisfying $\alpha(a) \le \beta(f(a)), \forall a \in A$.

Fuz has the topoi properties such as

(i) Equalizers, finite product and exponential exist in Fuz.

(ii) Terminal object exists in Fuz.

(iii) Fuz has no SC.

So Fuz is not a topos. However, category Fuz has two special objects:

(ii)' Middle object exists in Fuz, i.e., there is an object (I,α_I) such that for any object (A,α) in Fuz, there is exactly one morphism $f: A \longrightarrow I$ satisfying $\alpha_I(f(a))=\alpha(a), \forall a \in A$. (Where $I=(0, 1]$, $\alpha_I(\lambda)=\lambda$).

(iii)' There is an object (J,α_J) and a morphism $1_I: I \longrightarrow J$ such that for any monomorphism $m: (A',\alpha') \longrightarrow (A,\alpha)$ there is exactly one morphism α_m from (A,α) to (J,α_J) satisfying (1) $\alpha_m(a) \le \alpha(a), \forall a \in A$; (2) The figure 2 is a pullback.

$$\begin{array}{ccc} (A',\alpha') & \xrightarrow{\ \alpha'\ } & (I,\alpha_I) \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle 1_I} \\ (A,\alpha) & \xrightarrow{\ \alpha_m\ } & (J,\alpha_J) \end{array}$$

Figure 2

where $J=[0, 1]$, $\alpha_J(\lambda)=1$, $1_I(\lambda)=\lambda$, $\alpha_m(a)=\begin{cases}\alpha'(a'), & \text{if } a=m(a')\in m(A') \\ 0 & , \text{else}\end{cases}$

the α_m can be seen as the membership function of fuzzy set

(A',α'). So α_m is called as MF of $\{(A',\alpha'),m\}$.

Category Fuz safisfying (i), (ii)′ and (iii)′ is called as a WTopos.

2. Category M-Fuz is also a WTopos

Let $M=(M,*,e)$ be a monoid and A be any set. (A,λ) is called as a M-set if (1) $\lambda: M\times A \longrightarrow A$ is a mapping, (2) $\lambda(e,a)=a$, (3) $\lambda(m,\lambda(n,a))=\lambda(m*n,a), \forall a\in A, \forall m,n\in M$. Category M-Set of M-Sets is a topos.

Let (A,α) be an object of category Fuz, (A,α,λ) is called as a M-fuzzy set if (i) (A,λ) be a M-set, (ii) $\alpha(\lambda(m,a))=\alpha(a), \forall a\in A, m\in M$.

Let M-Fuz be a category. Its object is M-fuzzy set (A,α,λ); a morphism from (A,α,λ) to (B,β,μ) be a mapping $f: A \longrightarrow B$ satisfying (1) $\mu(m,f(a))=f(\lambda(m,a))$, (2) $\beta(\mu(m,f(a)))\geq\alpha(\lambda(m,a)), \forall a\in A, m\in M$.

Theorem 1. Category M-Fuz has all topos properties except one for it has no SC.

Let $\lambda_I(m,h)=h, \forall h\in I$, then for any M-fuzzy set (A,α,λ) there is exactly one morphism f from (A,α,λ) to (I,α_I,λ_I) such that $f(\lambda(m,a))=\alpha(\lambda(m,a))$. (I,α_I,λ_I) be a middle object of category **M-Fuz**.

Let $\lambda_J(m,h)=h, \forall m\in M, h\in J$. Then we have

Theorem 2. For any monomorphism $\phi:(A',\alpha',\lambda') \longrightarrow (A,\alpha,\lambda)$ there is exactly one morphism α_ϕ satisfying

(i) $\alpha_\phi(\lambda(m,a))\leq\alpha(\lambda(m,a))$;

(ii) The figure 3 is a pullback.

$$\begin{array}{ccc} (A',\alpha',\lambda') & \xrightarrow{\alpha'} & (I,\alpha_I,\lambda_I) \\ \phi \downarrow & & \downarrow 1_I \\ (A,\alpha,\lambda) & \xrightarrow{\alpha_\phi} & (J,\alpha_J,\lambda_J) \end{array}$$

Figure 3

Where $\alpha_\phi(\lambda(m,a))=\begin{cases} \alpha'(\lambda(m,a')), & \text{if } a=\phi(a')\in\phi(A') \\ 0 & \text{, else} \end{cases}$

α_ϕ is called as MMF of $\{(A',\alpha',\lambda'),\phi\}$.

We know from theorem 2 and definition of WTopos that category M-Fuz is also a WTops.

3. Category M-Fuz Has Power Objects

A category Ξ with finite product is said to have power objects if to each Ξ-object A there Ξ-object $\mathbb{P}(A)$, $\mathbb{E}_A$ and a monomorphism $\psi:\mathbb{E}_A \longrightarrow \mathbb{P}(A)\times A$, such that for any Ξ-object B and "relation" $r:R\longrightarrow B\times A$, there is exactly one Ξ-morphism $f_r:B\longrightarrow \mathbb{P}(A)$ for which there is a pullback in Ξ of the figure 4.

$$\begin{array}{ccc} R & \xrightarrow{\ r\ } & B\times A \\ \downarrow & & \downarrow f_r\times 1_A \\ \mathbb{E}_A & \xrightarrow{\ \psi\ } & \mathbb{P}(A)\times A \end{array}$$

Figure 4

Let (A,α,λ) and (B,β,μ) be two M-fuzzy sets, a "relation" from (A,α,λ) to (B,β,μ) is a M-fuzzy set (R,α_R,λ_R) satisfying $\lambda_R(m,(b,a))=(\mu(m,b),\lambda(m,a))$ and $\alpha_R(b,\lambda(m,a))=\alpha_R(\mu(m,b),a)\le \mathrm{Min}\{\alpha(a),\beta(b)\}$. Then we have

Theorem 3. In category M-Fuz, for any object (A,α,λ), there are object $(\mathbb{F}_A,\alpha_F,\lambda_F)$, $(\mathbb{E}_A,\alpha_E,\lambda_E)$ and monomorphism $id:(\mathbb{E}_A,\alpha_E,\lambda_E)\longrightarrow(\mathbb{F}_A,\alpha_F,\lambda_F)\times(A,\alpha,\lambda)$, such that for any object (B,β,μ) and "relation" (R,α_R,λ_R), there exactly one morphism $f_r:(B,\beta,\mu)\longrightarrow(\mathbb{F}_A,\alpha_F,\lambda_F)$ satisfying (1). $f(\mu(n,b))(\lambda(m,a))\le\alpha_R(\mu(n,b),\lambda(m,a))$; (2). The figure 5 is a pullback.

$$\begin{array}{ccc} (R,\alpha_R,\lambda_R) & \xrightarrow{\ r\ } & (B,\beta,\mu)\times(A,\alpha,\lambda) \\ \downarrow & & \downarrow f_r\times 1_A \\ (\mathbb{E}_A,\alpha_E,\lambda_E) & \xrightarrow{\ id\ } & (\mathbb{F}_A,\alpha_F,\lambda_F)\times(A,\alpha,\lambda) \end{array}$$

Figure 5

Where $\mathbb{F}_A=\{f\mid f:A\longrightarrow[0,\ 1]$ is a mapping satisfying $(*)\}$

(1) $f(a)\le\alpha(a)$, (2). $f(\lambda(m,a))\le f(a),\forall a\in A, m\in M$ (*)

$\lambda_F(m,f)(a)=f(\lambda(m,a)),\forall m\in M, a\in A, f\in\mathbb{F}_A$; $\alpha_F(f)=1,\forall f\in\mathbb{F}_A$

$\mathbb{E}_A=\{\langle f,a\rangle\mid f\in\mathbb{F}_A,\ a\in A$ and $f(a)\neq 0\}$

$$\lambda_E(m,\langle f,a\rangle)=\langle\lambda_F(m,f),a\rangle,\ \alpha_E(\langle f,a\rangle)=f(a)$$

$$r(b,a)=(b,a),\forall(b,a)\in R,\ id(\langle f,a\rangle)=\langle f,a\rangle,\forall\langle f,a\rangle\in \mathbb{E}_A$$

$$f_r(b)(a)=\begin{cases}\alpha_R(b,a), & \text{if } (b,a)\in R\\ 0, & \text{if } (b,a)\notin R\end{cases}$$

We know from theorem 3 and definition of power object that category M-Fuz has power object.

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