A Note on Fuzzy Volterra Integral Equations

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In this paper existence theorems for certain Volterra integral equations involving fuzzy set valued mappings (whose values are normal, convex, upper semicontinuous and compactly supported fuzzy sets in $\mathbb{R}^n$) are obtained.

1. Introduction : It is known in the literature (see [1] and [4]) that the Volterra integral equation has a solution under suitable assumptions. The purpose of this note is to generalize such existence theorems to fuzzy-valued mappings.

2. Preliminaries : By $\mathbb{P}_K(\mathbb{R}^n)$ we denote the family of all non-empty compact convex subsets of $\mathbb{R}^n$. Addition and scalar multiplication in $\mathbb{P}_K(\mathbb{R}^n)$ are defined as usual. $\bar{U}$ denotes the closure of U where U is contained in $\mathbb{R}^n$. Let T be the closed and bounded interval $[a,b] \subseteq \mathbb{R}$.

Define $E^n = \left\{ u : \mathbb{R}^n \longrightarrow [0,1] \text{ satisfying conditions (a) to (d) below} \right\}$

(a) u is normal i.e. $\exists\, x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$;

(b) u is fuzzy convex;

(c) u is uppersemicontinuous;

(d) $[u]^0 = \overline{\left\{ x \in \mathbb{R}^n \,/\, u(x) > 0 \right\}}$ is compact.

For $0 < \alpha \le 1$ denote $[u]^\alpha = \left\{ x \in \mathbb{R}^n \,/\, u(x) \ge \alpha \right\}$. We have from (a) to (d) that the α-level sets $[u]^\alpha \in \mathbb{P}_K(\mathbb{R}^n)$ for all $0 \le \alpha \le 1$.

If $g: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a function then using Zadeh's extension principle we can extend g to $E^n \times E^n \longrightarrow E^n$ by the equation $g(u,v)(z) = \sup_{z=g(x,y)} \min (u(x),v(y))$. It is well known that $[g(u,v)]^\alpha = g([u]^\alpha,[v]^\alpha)$ for all $u,v \in E^n$, $0 \le \alpha \le 1$ and g continuous (see [5]). For addition the above equation gives $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$. The real numbers can be embedded in E^1 by the rule $c \longrightarrow \hat{c}(t) = \begin{cases} 1 & \text{for } t=c \\ 0 & \text{elsewhere} \end{cases}$. We can also generalize multiplication by a real number and for any real number c we get $[c\, u]^\alpha = c\, [u]^\alpha$ where $0 \le \alpha \le 1$ and $u \in E^n$.

Let $D : E^n \times E^n \longrightarrow \mathbb{R}^+ \cup \{0\}$ be defined by

$$D(u,v) = \sup_{0 \le \alpha \le 1} H([u]^\alpha, [v]^\alpha)$$

where H is the Hausdorff metric defined in $\mathbb{P}_K(\mathbb{R}^n)$. Then D is a metric on E^n. Further (E^n,D) is a complete metric space (see [2] and [6]). Also $D(u+w,v+w) = D(u,v)$ for every $u,v \in E^n$. Furthermore , $D(\lambda u,\lambda v) = |\lambda|\, D(u,v)$ for every $u,v,w \in E^n$ and $\lambda \in \mathbb{R}$.

It can be proved straightaway that $D(u+v,w+z) \le D(u,w) + D(v,z)$ where $u,v,w,z \in E^n$. (The proof is based on the observation that $H(A_1 + A_2, B_1 + B_2) \le H(A_1,B_1) + H(A_2,B_2)$ where H is the Hausdorff metric on $\mathbb{P}_K(\mathbb{R}^n)$ induced by the norm in $\mathbb{R}^n$.)

Definition 2.1 : (see [3]) A mapping $F : T \longrightarrow E^n$ is strongly measurable if for all $\alpha \in [0,1]$ the set-valued map $F_\alpha : T \longrightarrow \mathbb{P}_K(\mathbb{R}^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lesbegue measurable when $\mathbb{P}_K(\mathbb{R}^n)$ has the topology induced by the Hausdorff metric H.

Definition 2.2 : (see [3]) A map $F : T \longrightarrow E^n$ is said to be integrably bounded if there is an integrable function h such that $\| x \| \le h(t)$ for every $x \in F_0(t)$.

It was proved by Puri and Ralescu [6] that a strongly measurable and integrably bounded mapping $F : T \longrightarrow E^n$ is integrable. $\left[\text{i.e. } \int_T F(t)\, dt \in E^n \right]$.

For the proofs of the following theorems [3] may be referred.

Theorem 2.1 : If $F : T \longrightarrow E^n$ is continuous then it is integrable.

Theorem 2.2 : Let $F, G : T \longrightarrow E^n$ be integrable and $\lambda \in \mathbb{R}$.

Then

(i) $\int (F+G) = \int F + \int G$

(ii) $\int \lambda F = \lambda \int F$

(iii) $D(F,G)$ is integrable

(iv) $D\left[\int F, \int G \right] \le \int D(F, G)$

3. Existence theorems :

Theorem 3.1 : Consider the following non-linear fuzzy-valued Volterra integral equation

$$x(t) = f(t) + \int_0^t g(t,s,x(s))\, ds \qquad (1)$$

We make the following assumptions:-

Let a, b and L be positive numbers and for some fixed $\alpha \in (0,1)$ define $c = \alpha / L$.

Suppose (i) $f : [0,a] \longrightarrow E^n$ continuous.

(ii) $g : U \longrightarrow E^n$ continuous where $U = \{(t,s,x) \,/\, 0 \le s \le t \le a,$ $x \in E^n$ and $D(x,f(t)) \le b\}$.

(iii) g satisfies Lipschitz condition with respect to x on U i.e $D(\, g(t,s,x),\ g(t,s,y)\,) \le L\, D(x,y)$ if $(t,s,x),(t,s,y) \in U$. If $M = \max_U D(\, g(t,s,x),\ \hat{0}\,)$, then there is a unique solution of (1) on $[0,T]$ where $T = \min[\, a,\ b/M,\ c\,]$.

Proof : Let $\mathbb{C}$ be the space of continuous functions from $[0,T]$ into (E^n,D) with $H_1(\psi,f) \le b$ i.e. $\mathbb{C} = \{\psi \,/\, \psi : [0,T] \longrightarrow E^n$ continuous and $H_1(\psi,f) \le b\}$ where $H_1(\psi,f) = \sup_{0\le t\le T} D(\psi(t),f(t))$ where D is as defined earlier.

Define an operator $A : \mathbb{C} \longrightarrow \mathbb{C}$ by

$$A\psi(t) = f(t) + \int_0^t g(t,s,\psi(s))\, ds$$

To prove that $A : \mathbb{C} \longrightarrow \mathbb{C}$ we have to prove that $A\psi$ is continuous whenever $\psi \in \mathbb{C}$ and that $H_1(A\psi,f) \le b$.

Consider $D(A\psi(t+h),\ A\psi(t))$

$$= D\left[f(t+h) + \int_0^{t+h} g(t+h,s,\psi(s))\, ds,\ f(t) + \int_0^t g(t,s,\psi(s))\, ds\right]$$

$$\le D(f(t+h),\ f(t)) + D\left[\int_0^{t+h} g(t+h,s,\psi(s))\, ds,\ \int_0^t g(t,s,\psi(s))\, ds\right]$$

$$\le \frac{\epsilon}{2} + D\left[\int_0^t g(t+h,s,\psi(s))\, ds,\ \int_0^t g(t,s,\psi(s))\, ds\right] +$$

$$D\left[\int_t^{t+h} g(t+h,s,\psi(s))\, ds,\ \hat{0}\right]$$

(The first term on the right hand side is less than $\varepsilon/2$ as f is continuoous at any $t \in [0,T]$)

$$\le \frac{\varepsilon}{2} + \int_0^t D(g(t+h,s,\psi(s)),g(t,s,\psi(s)))\,ds + \int_t^{t+h} D(g(t+h,s,\psi(s)),\hat{0})\,ds \quad .....(I)$$

Clearly the right hand side of (I) tends to zero as $h \longrightarrow 0$.

So $A\psi$ is continuous.

$$\begin{aligned}
\text{Consider } H_1(A\psi,f) &= \sup_{0\le t\le T} D(A\psi(t),f(t)) \\
&= \sup_{0\le t\le T} D(f(t) + \int_0^t g(t,s,\psi(s))ds, f(t)) \\
&= \sup_{0\le t\le T} D(\int_0^t g(t,s,\psi(s))ds,\hat{0}) \\
&\le \sup_{0\le t\le T} \int_0^t D(g(t,s,\psi(s),\hat{0})ds \\
&\le M\,T \le b
\end{aligned}$$

So $A\psi \in \mathbb{C}$. A maps $\mathbb{C}$ into itself.

We show that $\mathbb{C}$ is a closed subset of $C([0,T],E^n)$, a complete metric space with the metric H_1 (see [3]).

Let (ψ_n) be a sequence in $\mathbb{C}$ converging to ψ in $C([0,T],E^n)$. Consider $H_1(\psi,f) \le H_1(\psi_n,\psi) + H_1(\psi_n,f)$

$$\le \varepsilon + b$$

for sufficiently large n and all positive ε. So $\psi \in \mathbb{C}$. This implies that $\mathbb{C}$ is a closed subset of $C([0,T],E^n)$. Therefore $\mathbb{C}$ is a complete metric space.

We prove that A is a contraction mapping.

For $\phi,\psi \in \mathbb{C}$, $H_1(A\phi,A\psi) = \sup_{0\le t\le T} D(A\phi(t),A\psi(t))$

$$= \sup_{0\le t\le T} D\left[f(t)+\int_0^t g(t,s,\phi(s))ds,\ f(t)+\int_0^t g(t,s,\psi(s))ds\right]$$

$$= \sup_{0\le t\le T} D\left[\int_0^t g(t,s,\phi(s))ds,\ \int_0^t g(t,s,\psi(s))ds\right]$$

$$\le \sup_{0\le t\le T} \int_0^t D\left[g(t,s,\phi(s)),g(t,s,\psi(s))\right]ds$$

$$\le \sup_{0\le t\le T} \int_0^t L\ D(\phi(s),\psi(s))ds \text{ in view of (iii)}$$

$$\le T\ L\ H_1(\phi,\psi)$$

$$\le c\ L\ H_1(\phi,\psi) = \alpha\ H_1(\phi,\psi) \text{ where } \alpha \in (0,1).$$

So $A : \mathbb{C} \longrightarrow \mathbb{C}$ is a contraction map. Since $\mathbb{C}$ is a complete metric space and A is a contracting self-map on $\mathbb{C}$, it has a unique fixed point $x \in \mathbb{C}$. This fixed point is the required unique solution to the equation (1).

We consider now the fuzzy Volterra integral equation

$$x(t) = \lambda \int_a^t K(t,s)\ x(s)\ ds + \phi(t) \qquad (2)$$

Theorem 3.2 : Suppose $K(t,s) : [a,b] \times [a,b] \longrightarrow \mathbb{R}$ and $\phi : [a,b] \longrightarrow E^n$ be given continuous functions and λ an arbitrary parameter. If $|K(t,s)| \le M$ for all $a \le t,s \le b$ then the equation (2) has a unique fuzzy-valued solution.

Proof : Let T be a closed interval [a,b]. Define G on $C(T,E^n)$ by

$$Gx(t) = \lambda \int_a^t K(t,s)\ x(s)\ ds + \phi(t)$$

It is easy to see that Gx is continuous whenever $x \in C(T,E^n)$. So $Gx \in C(T,E^n)$ whenever $x \in C(T,E^n)$.

Now let $x_1, x_2 \in C(T,E^n)$

Consider $D(Gx_1(t), Gx_2(t))$ which is

$$D\left[\lambda \int_a^t K(t,s)x_1(s)\,ds + \phi(t), \lambda \int_a^t K(t,s)x_2(s)\,ds + \phi(t)\right]$$

$$= |\lambda|\ D\left[\int_a^t K(t,s)x_1(s)\,ds, \int_a^t K(t,s)x_2(s)\,ds\right]$$

$$\le |\lambda| \int_a^t D\left[K(t,s)x_1(s), K(t,s)x_2(s)\right] ds$$

$$\le |\lambda|\ M \int_a^t D\left[x_1(s), x_2(s)\right] ds$$

$$\le |\lambda|\ M\ (t-a)\ H_1(x_1, x_2)$$

Similarly $D\left[G^2x_1(t), G^2x_2(t)\right] \le \dfrac{|\lambda|^2 M^2 (t-a)^2}{2!} H_1(x_1, x_2)$

So inductively for all n,

$$D\left[G^nx_1(t), G^nx_2(t)\right] \le \frac{|\lambda|^n M^n (b-a)^n}{n!} H_1(x_1, x_2)$$

$$\Longrightarrow \qquad H_1\left[G^nx_1, G^nx_2\right] \le \frac{|\lambda|^n M^n (b-a)^n}{n!} H_1(x_1, x_2)$$

So given any λ, we can always choose n large enough to make $\dfrac{|\lambda|^n M^n (b-a)^n}{n!} < 1$. Thus G^n is a contraction mapping whenever n is sufficiently large. So by Banach's contraction principle, the Fuzzy Volterra integral equation (2) has a unique solution.

Example 3.1 : Let $K(t,s) = \sin(t)\cos(s)$ and $f : [2,3] \longrightarrow E^1$ be a function defined by

$$f(t)(x) = \begin{cases} xt & \text{if } x \in [0,(1/t)] \\ 1 & \text{if } x \in [(1/t),1-(1/t)] \\ (1-x)t & \text{if } x \in [1-(1/t),1] \\ 0 & \text{otherwise} \end{cases}$$

Then $[f(t)]^{\alpha} = [(\alpha/t)\ ,\ 1-(\alpha/t)]$ for $0 < \alpha \le 1$ and $[f(t)]^0 = [0,1]$. It is easy to check that the fuzzy function $f : [2,3] \longrightarrow E^1$ defined above is a continuous fuzzy function. Then our existence theorem states that the fuzzy Volterra equation

$$x(t) = \lambda \int_2^t K(t,s)\, x(s)\, ds + f(t)$$

has a unique fuzzy-valued solution for any λ.

One may also choose $f(t) = \hat{r}(t) + A$ as considered by Kaleva (see example 5.1 [3]) where $A \in E^n$ is fixed.

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