AN EXTENDED FORM OF THE EXTENSION PRINCIPLE FOR FUZZY SETS

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ABSTRACT

In this paper, we propose an extended form of the extension principle for fuzzy sets and discuss its properties.

Keywords: Extension principle, resolution of identity, E− extension principle.

1. INTRODUCTION

The extension principle, first proposed by L.A.Zadeh ,is an important principle for fuzzy set theory and is particularly useful in applications which call for an extension of the domain of a function. It provides a natural way for extending the domain of a mapping on a set U to the collection of all fuzzy subsets of U. In this paper, we propose an extended form of the extension principle, E− extension principle, and discuss its properties.

We recall some basic concepts and notations of fuzzy set theory which will be used in the sequel.

The collection of all the fuzzy subsets of a set X is denoted by F(X). For $\alpha \in [0,1]$, the α −level set A_α and strong α− level set $A_{\underset{\cdot}{\alpha}}$ of $A \in F(X)$ are defined by

$$A_\alpha = \{ x \in X; A(x) \geq \alpha \}$$

and

$$A_{\underset{\cdot}{\alpha}} = \{ x \in X; A(x) > \alpha \}$$

The resolution of identity[1] for fuzzy sets shows that

$$A = \bigcup_{\alpha \in [0,1]} \alpha A_{\alpha} = \bigcup_{\alpha \in [0,1]} \alpha A_{\underset{\cdot}{\alpha}} \tag{1.1}$$

If f: X → Y is a mapping, A ∈ F(X), then the fuzzy set f(A) on Y is defined, via the extension principle, by

$$\text{for every y in Y, } f(A)(y) = \bigvee_{x \in f^{-1}(y)} A(x) \tag{1.2}$$

for B ∈ F(Y), $f^{-1}(B)$ is defined by

$$f^{-1}(B) \in F(X), \textit{ and for every x in X, } f^{-1}(B)(x) = B(f(x)) \tag{1.3}$$

2. E–EXTENSION PRINCIPLE

Definition 2.1 If R ∈ F(X × Y) is a fuzzy relation on X and Y, A ∈ F(X), then the fuzzy subset R(A) on Y is defined, via the E–extension principle, by

$$\text{for every y in Y, } R(A)(y) = \bigvee_{x \in X} (R(x,y) \wedge A(x)) \tag{2.1}$$

for B ∈ F(Y), $R^{-1}(B)$ is defined by

$$R^{-1}(B) \in F(X), \text{ and for every x in X, } R^{-1}(B)(x) = \bigvee_{y \in Y} (R(x,y) \wedge B(y)) \tag{2.2}$$

If f:X → Y is a mapping, the relation R_f on X × Y is called the relation defined by f, where

$$R_f(x,y) = \begin{cases} 1, & y = f(x) \\ 0, & y \neq f(x) \end{cases}$$

The following theorem shows that the E– extension principle is anextended form of the extension principle.

Theorem 2.2 If f:X→Y is a mapping, A ∈ F(X), B ∈ F(Y), then

$$R_f(A) = f(A) \tag{2.3}$$

$$R_f^{-1}(B) = f^{-1}(B) \tag{2.4}$$

Proof: For every y in Y,

$$R_f(A)(y) = \bigvee_{x\in X}(R_f(x,y)\wedge A(x))$$

$$=(\bigvee_{x\in f^{-1}(y)}(R_f(x,y)\wedge A(x)))\vee(\bigvee_{x\in X-f^{-1}(y)}(R_f(x,y)\wedge A(x)))$$

$$=\bigvee_{x\in f^{-1}(y)} A(x) = f(A)(y)$$

For every x in X,

$$R_f^{-1}(B)(x) = \bigvee_{y\in Y}(R_f(x,y)\wedge B(y)) = R_f(x,f(x))\wedge B(f(x))$$

$$= B(f(x)) = f^{-1}(B)(x).$$

This complete the proof.

Theorem 2.3 If $R\in F(X\times Y)$, $A_i\in F(X)$ $(i\in I)$, $B_j\in F(Y)$ $(j\in J)$, then

(i) $R(\Phi)=\Phi$, $R^{-1}(\Phi)=\Phi$.

(ii) $R(\bigcup_{i\in I}A_i)=\bigcup_{i\in I}R(A_i)$

(iii) $R^{-1}(\bigcup_{j\in J}B_j)=\bigcup_{j\in J}R^{-1}(B_j)$

(iv) $R(\bigcap_{i\in I}A_i)\subseteq\bigcap_{i\in I}R(A_i)$

(v) $R^{-1}(\bigcap_{j\in J}B_j)\subseteq\bigcap_{j\in J}R^{-1}(B_j)$.

Proof:

(i) Trivial.

(ii) For every y in Y, $R(\bigcup_{i\in I}A_i)(y)=\bigvee_{x\in X}(R(x,y)\wedge\bigcup_{i\in I}A_i(x))$

$$=\bigvee_{x\in X}(R(x,y)\wedge(\bigvee_{i\in I}A_i(x)))=\bigvee_{x\in X}\bigvee_{i\in I}(R(x,y)\wedge A_i(x))$$

$$=\bigvee_{i\in I}\bigvee_{x\in X}(R(x,y)\wedge A_i(x))=\bigvee_{i\in I}R(A_i)(y)=\bigcup_{i\in I}R(A_i)(y).$$

(iii) Similar to the proof of (ii).

(iv) If $A_1,A_2\in F(X)$, $A_1\subseteq A_2$, by (ii),

$$R(A_2)=R(A_1\bigcup A_2)=R(A_1)\bigcup R(A_2)$$

that is $R(A_1)\subseteq R(A_2)$. So $R(\bigcap_{i\in I}A_i)\subseteq R(A_i)$ $(i\in I)$ and hence

$$R(\bigcap_{i\in I}A_i)\subseteq\bigcap_{i\in I}R(A_i).$$

(v) Similar to the proof of (iv).

Theorem 2.4 If $R \in F(X \times Y)$, $A \in F(X)$, then

$$R(A) = \bigcup_{\alpha \in [0,1]} \alpha . R(A_\alpha) \tag{2.5}$$

$$R(A) = \bigcup_{\alpha \in [0,1]} \alpha . R(A_{\underset{\cdot}{\alpha}}) \tag{2.6}$$

Proof. For every y in Y, $\bigcup_{\alpha \in [0,1]} \alpha\, R(A_\alpha)(y) = \bigvee_{\alpha \in [0,1]} \alpha\, R(A_\alpha)(y)$

$$= \bigvee_{\alpha \in [0,1]} (\alpha \wedge (\bigvee_{x \in X} (R(x,y) \wedge A_\alpha(x))))$$

$$= \bigvee_{\alpha \in [0,1]} \bigvee_{x \in X} (\alpha \wedge R(x,y) \wedge A_\alpha(x))$$

$$= \bigvee_{x \in X} \bigvee_{\alpha \in [0,1]} (\alpha \wedge R(x,y) \wedge A_\alpha(x))$$

By the resolution of identity,

$$\bigvee_{\alpha \in [0,1]} (\alpha \wedge R(x,y) \wedge A_\alpha(x)) = R(x,y) \wedge (\bigvee_{\alpha \in [0,1]} (\alpha \wedge A_\alpha(x))) = R(x,y) \wedge A(x),$$

So we have $\bigcup_{\alpha \in [0,1]} \alpha R(A_\alpha)(y) = \bigvee_{x \in X} (R(x,y) \wedge A(x)) = R(A)(y)$.

Eq.(2.6) can be proved similarly.

Remark. For a crisp subset A, R(A) is not necessarily crisp.

Theorem 2.5 If $R \in F(X \times Y)$, $A \in F(X)$, $\alpha \in (0,1)$, then

$$R(A)_{\underset{\cdot}{\alpha}} \subseteq R(A_\alpha)_{\underset{\cdot}{\alpha}} \subseteq R(A_\alpha)_\alpha \subseteq R(A)_\alpha . \tag{2.7}$$

Proof: For every y in Y, if $y \bar{\in} R(A_\alpha)_{\underset{\cdot}{\alpha}}$, then

$$R(A_\alpha)(y) = \bigvee_{x \in X} (R(x,y) \wedge A_\alpha(x)) = \bigvee_{x \in A_\alpha} R(x,y) < \alpha,$$

So, $\bigvee_{x \in A_\alpha} (R(x,y) \wedge A(x)) < \bigvee_{x \in A_\alpha} R(x,y) < \alpha$,

$$\bigvee_{x \bar{\in} A_\alpha} (R(x,y) \wedge A(x)) \leqslant \bigvee_{x \bar{\in} A_\alpha} (R(x,y) \wedge \alpha) < \alpha,$$

and hence,

$$R(A)(y) = \bigvee_{x \in X} (R(x,y) \wedge A(x))$$

$$= (\bigvee_{x \in A_\alpha} (R(x,y) \wedge A(x))) \vee (\bigvee_{x \bar{\in} A_\alpha} (R(x,y) \wedge A(x))) < \alpha ,$$

So, $y \bar{\in} R(A)_{\underset{\cdot}{\alpha}}$. That is $R(A)_{\underset{\cdot}{\alpha}} \subseteq R(A_\alpha)_{\underset{\cdot}{\alpha}}$.

For every y in Y, if $y \in R(A_\alpha)_\alpha$, then

$$R(A_\alpha)(y) = \bigvee_{x \in X} (R(x,y) \wedge A_\alpha(x)) = \bigvee_{x \in A_\alpha} R(x,y) > \alpha$$

and hence,

$$R(A)(y) = \bigvee_{x \in X} (R(x,y) \wedge A(x)) > \bigvee_{x \in A_\alpha} (R(x,y) \wedge A(x)) > \bigvee_{x \in A_\alpha} (R(x,y) \wedge \alpha)$$

$$= \alpha \wedge \Big(\bigvee_{x \in A_\alpha} R(x,y)\Big) = \alpha,$$

that is $y \in R(A)_\alpha$ and hence $R(A_\alpha)_\alpha \subseteq R(A)_\alpha$, this complete the proof.

Theorem 2.6 If $R \in F(X \times Y)$, $A \in F(X)$, $\alpha \in (0,1)$, then

$$R(A)_{\underset{\cdot}{\alpha}} \subseteq R(A_{\underset{\cdot}{\alpha}})_{\underset{\cdot}{\alpha}} \subseteq R(A_{\underset{\cdot}{\alpha}})_\alpha \subseteq R(A)_\alpha . \tag{2.8}$$

The proof is similar to that of Theorem 2.5.

Corollary 2.7 For $R \in F(X \times Y)$, $A \in F(X)$, $\alpha \in (0,1)$,

$$R(A)_{\underset{\cdot}{\alpha}} \subseteq R(A_{\underset{\cdot}{\alpha}})_{\underset{\cdot}{\alpha}} \subseteq R(A_\alpha)_{\underset{\cdot}{\alpha}} \subseteq R(A_\alpha)_\alpha \subseteq R(A)_\alpha . \tag{2.9}$$

$$R(A)_{\underset{\cdot}{\alpha}} \subseteq R(A_{\underset{\cdot}{\alpha}})_{\underset{\cdot}{\alpha}} \subseteq R(A_{\underset{\cdot}{\alpha}})_\alpha \subseteq R(A_\alpha)_\alpha \subseteq R(A)_\alpha . \tag{2.10}$$

Theorem 2.8 For $R \in F(X \times Y)$, $A \in F(X)$, $\alpha \in (0,1)$, if for every y in Y, $\bigvee_{x \in X - A_\alpha} (R(x,y) \wedge A(x))$ is attained, then $R(A_\alpha)_\alpha = R(A)_\alpha$

Proof: By Eq.(2.7), it is sufficient to proof that $R(A)_\alpha \subseteq R(A_\alpha)_\alpha$.

If $y \overline{\in} R(A_\alpha)_\alpha$, then

$$R(A_\alpha)(y) = \bigvee_{x \in X} (R(x,y) \wedge A_\alpha(x)) = \bigvee_{x \in A_\alpha} R(x,y) < \alpha$$

and hence

$$\bigvee_{x \in A_\alpha} (R(x,y) \wedge A(x)) < \alpha$$

Since $\bigvee_{x \in X - A_\alpha} (R(x,y) \wedge A(x))$ is attained, there exist $x_0 \in X - A_\alpha$ such that

$$\bigvee_{x \in X - A_\alpha} (R(x,y) \wedge A(x)) = R(x_0,y) \wedge A(x_0)$$

So, $\bigvee_{x \in X - A_\alpha} (R(x,y) \wedge A(x)) < A(x_0) < \alpha$

and hence

$$R(A)(y)=\bigvee_{x\in X}(R(x,y)\wedge A(x))=(\bigvee_{x\in A_\alpha}(R(x,y)\wedge A(x)))\vee(\bigvee_{x\in X-A_\alpha}(R(x,y)\wedge A(x)))<\alpha,$$

that is $y\overline{\in} R(A)_\alpha$. This complete the proof.

Similar to the Extension principle, we can know the other properties about R and R^{-1}.

REFERENCE

〔1〕 L.A.Zadeh, The concept of linguistic variable and its application to approximate reasoning, Inform. Sci. 8(1975), 199–249, 301–357; 9(1975), 43–80.

〔2〕 H.T.Nguyen, A note on the extension principle for fuzzy sets, Journal of Math. Anal. and Appl., 64(1978), 369–380.

〔3〕 D. Dubois and H. Prade, Fuzzy Sets and Systems: Theory and Applications, Academic Press, New York, 1980.