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Toposes of (totally) fuzzy sets

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Abstract

The aim of this Note is to briefly present the toposes of fuzzy sets.[1)]

As it is well-known, the organization of fuzzy sets as toposes is fundamental in the solving the problem of rebuilding of some classical mathematical concepts in the framework created by fuzzy sets, as well as, in the construction of the local theories of fuzzy sets.

In [13] an overwiev about the categorical aspects of fuzzy sets is given. Here some recent results, on the same topics are presented.

1.Introduction

The evolution [2)] of the theory of fuzzy sets could be divided into two steps:

a) the "fuzzyfication" of the membership relation, the notion of fuzzy set (more exactly: fuzzy subset of a given reference set) being obtained. It is followed naturally by the definition of fuzzy function. From categorical point of view this type of fuzzyfication has been studied by J. A. Goguen [9], D.Ponasse [11], J. L. Bell [1], M. Eytan [8], J. C.Carrega [4].

b) a great advance in the theory of fuzzy sets is considered to be the fuzzyfication of the predicate of equality taking into consideration not only the "degree of membership" (of the elements x, y,... to a fuzzy set A), but also the "degree of indiscernability" (between x and y).

In this way is defined the notion of totally fuzzy set by G. Blanc [3] (and, as it is expected, the notion of totally fuzzy function). Some variations of this concept are studied by D. Higgs [10], J. L. Bell [1], J. and J. L. Coulon [6], D.Ponasse [11], S.Benkaddour [2], I.Tofan [14].

A somewhat different approach has been tried by U. Cerruti and U. Höhle [5] who introduced the category of undeterminate sets provided with a concrete equality.

2.Basical notions and results

Let H be a complete Heyting[3)] algebra.

a) H-fuzzy sets.

A H-fuzzy set (shortly fuzzy set) is a couple (A,α) where A is a non-empty set, and α is an arbitrary map from A to H.

A function[4)] f from (A,α) to (B,β) is a map from A to B such that, $\forall a \in A$, $\alpha(a) \leq \beta(f(a))$.

Considering H-fuzzy sets as objects and the above maps as morphisms (with the usually composition) we thus get a category denoted by Set V [9] (or VF [11]).

An alternative definition for morphisms is proposed in [8]: a function f from (A,α) to (B,β) is a map f from $A \times B$ to H such that the following properties are fulfilled:

i) $f(a,b) \leq \alpha(a) \wedge \beta(b)$;

ii) $V_b f(a,b) = \alpha(a)$;

iii) $f(a,b) \wedge f(a,b') = 0$ if $b \neq b'$.

The composition law is defined by $gf(a,c) = V_b(f(a,b) \wedge g(b,c))$ where $f{:}A \times B \longrightarrow H$, $g{:}B \times C \longrightarrow H$. One obtain the category Fuz H [1,4,8,14].

The following propositions are true [1,4,8,14]:

* The categories Set V and Fuz H are isomorphic.

* Fuz H (and then Set V) is an elementary topos if and only if H is a boolean algebra (and if and only if Fuz H is a boolean topos).

b) totally H-fuzzy sets.

A totally H-fuzzy set (shortly totally fuzzy set) is a couple (X,σ) where X is a non empty set and σ is a map from $X \times X$ to H verifying:

i) $\sigma(x,y) = \sigma(y,x)$;

ii) $\sigma(x,y) \wedge \sigma(y,z) \leq \sigma(x,z)$.

The functions from (X,σ) to (Y,τ) are the mappings from $X \times Y$ to H such that:

i) $f(x,y) \wedge \sigma(x,x') \leq f(x',y)$;

ii) $f(x,y) \wedge \tau(y,y') \leq f(x,y')$;

iii) $f(x,y) \wedge f(x,y') \leq \tau(y,y')$;

iv) $V_y f(x,y) = \sigma(x,x)$.

The identity function from (x,σ) to (x,σ) is σ.

The composition of functions f from (X,σ) to (Y,τ) and g from (Y,τ) to (Z,ρ) is defined by $gf(x,z) = V_y(f(x,y) \wedge g(y,z))$.

Considering totally H-fuzzy sets as objects, and the above functions as morphisms, we thus get the category S(H) [1,10,14].

We obtain:

* S(H) is an SG-Ens-topos[5)].

An alternative definition of morphisms is given by using the connection between the notion of function and the notion of relation: the functions from (X,σ) to (Y,τ) are the binary relations R from X to Y verifying:

i) $\forall\, x \in X$, $\{y \in Y, / xRy\} \neq \phi$;

ii) (xRy and x'Ry') $\Longrightarrow \sigma(x,x') \leq \tau(y,y')$;

iii) (xRy and $\sigma(x,x) \leq \tau(y,y')$) $\Longrightarrow$xRy'.

The identity function I from (X,σ) to (X,σ) is defined by $xIx' \Longleftrightarrow \sigma(x,x) = \sigma(x,x')$.

The composition of functions R from (X,σ) to (Y,τ) and S from (Y,τ) to (Z,ρ) is defined by $xSRz \Longleftrightarrow \exists y \in Y$ and $z' \in Z$ such that xRy and ySz with $\sigma(x,x) \leq \rho(z,z')$.

One obtain the category JTF [6,7,13] and the following results:

* S(H) is equivalent to a full subcategory of JTF, and if H is an antiordinal[6)] then S(H) is equivalent to JTF.

* JTF is an elementary topos if and only if H is an antiordinal.

The connection of the objects of JTF with the notion of presheaf on H leads to the category JTF^{00} [7] (which is a full subcategory of JTF and equivalent to JTF).

Another aspect of totally fuzzy sets is provided by constructing the category JID [2] isomorphic to JTF^{00}.

3. Concluding remarks

* Sumarizing the above results one obtains:

- for fuzzy sets :

- if H is a Heyting algebra, then we have the categories Set V $\cong$ Fuz H;

- if H is a boolean algebra, then we have the boolean toposes Set V $\cong$ Fuz H.

- for totally fuzzy sets:

- if H is a Heyting algebra, then we have the categories JTF $\simeq$ JTF^{00} $\cong$ JID and the SG-Ens-topos S(H) which is equivalent to a subcategory of JTF^{00};

- if H is an antiordinal then we have the SG-Ens-toposes S(H) $\simeq$ JTF $\simeq$ JTF^{00} $\cong$ JID.

* If (A,α) is a fuzzy set then the indiscernability function may be defined by $\sigma(a,b) = \alpha(a) \wedge \alpha(b)$ ($\sigma: A\times A \longrightarrow H$). Conversely if (X,σ) is a totally fuzzy set then the membership function may be defined by $\alpha(x) = \sigma(x,x)$ ($\alpha: A \longrightarrow H$).

*If H is an antiordinal then S(H), JTF, JTF^{00}, JID as SG-Ens-toposes, possess a natural number object, an object of integers, an object of real numbers.

* If H is a boolean algebra then the classifier subobject (Ω) of Set V and Fuz H has an internal structure of Boole algebra. Besides, if $\rceil : \Omega \longrightarrow \Omega$ is the characteristic morphism of the subobject $f: 1 \longrightarrow \Omega$ (the characteristic morphism of the subobject $0 \longrightarrow 1$ where 0 is the initial object of Set V, Fuz H) then $\rceil\rceil = 1_\Omega$ (the morphism $\rceil$ is called the negation morphism).

4.Notes

1) This paper is a version of a part of [14].

2) Every short history of fuzzy sets must include references to the Aristote (which had developped the basical ideas of the nowadays "possibility theory"), to the papers of H. Poincare, to the papers of H. Weyl (which had defined a concept very near to the future notion of fuzzy set), to L. Zadeh (which had formulated the basical notions and results of the theory of fuzzy sets and had succeeded to impose this theory) and to G. Moisil

(which had interpreted the fuzzy sets as an extension of a predicate in a multivalued logic).

3) The choice of Heyting algebras as value set of the membership degree may be connected with the remark that the classifier subobjects of an elementary topos has a structure of internal Heyting algebra.

4) This function is more a function between fuzzy sets than fuzzy functions.

5) i.e.:

i) the collection of subobjects of the final object 1 is a generating class, in other words, for every distinct morphisms $A \overset{g}{\underset{f}{\rightrightarrows}} B$ there is a subobject U⟶1, and a morphism h:U⟶A, such that $fh \neq gh$ - and-

ii) S(H) has arbitrarily coproducts of subobjects of 1, in other words, then exist the functors F^*:Ens⟶S(H) and F_*:S(H)⟶Ens, where Ens denote the topos of sets, such that F^* is left adjoint to F_* and F_* preserve finite limits. Let us notice that every SG-Ens-topos is a Grothendieck topos.

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