SOME MODAL TYPE OF OPERATORS IN INTUITIONISTIC FUZZY MODAL LOGIC

Part. I

Krassimir T. Atanassov

Math. Research Lab. - IPACT, P.O.Box 12, Sofia-1113, BULGARIA

The definition of the Intuitionistic Fuzzy Set (IFS) is the basis for defining the following parts of the Intuitionistic Fuzzy Logic (IFL): Intuitionistic Fuzzy Propositional Calculus (IFPC) [1], Intuitionistic Fuzzy Predicate Logic (IFPL) [2], Intuitionistic Fuzzy Modal Calculus (IFMC) [3], Temporal Intuitionistic Fuzzy Logic (TIFL) [4].

Firstly, we shall introduce some elements of IFPS and IFML.

To each proposition (in the classical sense) one can assign its truth value: truth - denoted by 1, or falsity - 0. In the case of fuzzy logics this truth value is a real number in the interval [0, 1] and can be called "truth degree" of a particular proposition. Here we add one more value - "falsity degree" - which will be in the interval [0, 1] as well. Thus one assigns to the proposition p two real numbers $\mu(p)$ and $\gamma(p)$ with the following constraint to hold:

$$\mu(p) + \gamma(p) \le 1.$$

Let this assignment be provided by an evaluation function V defined in such a way that:

$$V(p) = \langle \mu(p), \gamma(p) \rangle.$$

Hence the function V gives the truth and falsity degrees of the given propositions.

We assume that the evaluation function V assigns to the logical truth T: $V(T) = \langle 1, 0 \rangle$, and to the logical false F: $V(F) = \langle 0, 1 \rangle$.

The evaluation of the negation $\neg p$ of the proposition p will be defined through:

$$V(\neg p) = \langle \gamma(p), \mu(p) \rangle.$$

When $\gamma(p) = 1 - \mu(p)$, i.e.

$$V(p) = \langle \mu(p), 1 - \mu(p) \rangle,$$

for $\neg p$ we get:

$$V(\neg p) = \langle 1 - \mu(p), \mu(p) \rangle,$$

which coincides with the result for an ordinary fuzzy logic (see e.g [5,6]).

When the values V(p) and V(q) of the propositions p and q are known, the evaluation function V can be extended also for the operations "&", "$\vee$" through the definition:

$$V(p \,\&\, q) = \langle \min(\mu(p), \mu(q)), \max(\gamma(p), \gamma(q)) \rangle,$$

$$V(p \vee q) = \langle \max(\mu(p), \mu(q)), \min(\gamma(p), \gamma(q)) \rangle,$$

By analogy with the operations over IFSs it will be convenient to define for the propositions p, q $\in$ S:

$$\neg V(p) = V(\neg p),$$
$$V(p) \wedge V(q) = V(p \,\&\, q),$$
$$V(p) \vee V(q) = V(p \veebar q),$$

Depending on the way of defining the operation "$\supset$" different variants of IFPC can be obtained (see [1, 3]).

One possibility for the evaluation of the compound proposition $p \supset q$ is given by:

$$V(p \supset q) = \langle 1 - (1 - \mu(q)).sg(\mu(p) - \mu(q)),\ \gamma(q).sg(\mu(p) - \mu(q)).sg(\gamma(q) - \gamma(p))\rangle,$$

where:

$$sg(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \le 0. \end{cases}$$

and it is named sg-variant of an implication.

The other variant, named (max-min)-variant is defined by:

$$V(p \supset q) = \langle \max(\gamma(p), \mu(q)), \min(\mu(p), \gamma(q))\rangle.$$

For both cases of implication ($\supset$), let

$$V(p) \to V(q) = V(p \supset q).$$

For the needs of the discussion below following [1, 3] we shall define the notion of Intuitionistic Fuzzy Tautology (IFT):

A is an IFT iff if $V(A) = \langle a, b\rangle$, then $a \ge b$,

while

A is a standard tautology iff $V(A) = \langle 1, 0\rangle$.

For a proposition p, for which $V(p) = \langle a, b\rangle$, in [3] are defined the operators "necessity" and "possibility" by (cf. [7]):

$$V(\Box p) = \langle a, 1-a\rangle,$$
$$V(\Diamond p) = \langle 1-b, b\rangle.$$

Here, by analogy with the IFS-operators (see [8-10]) we shall define 8 new operators. Let p be a fixed proposition and α, β, Γ, ∂, φ, $\psi \in [0, 1]$. We define:

$$D_{\alpha}(p) = \langle a + \alpha.(1 - a - b),\ b + (1 - \alpha).(1 - a - b)\rangle,$$

$$F_{\alpha, \beta}(p) = \langle a + \alpha.(1 - a - b),\ b + \beta.(1 - a - b)\rangle, \text{ for } \alpha + \beta \le 1,$$

$$G_{\alpha, \beta}(p) = \langle \alpha.a,\ \beta.b\rangle,$$

$$H_{\alpha, \beta}(p) = \langle \alpha.a,\ b + \beta.(1 - a - b)\rangle,$$

$$H^{*}_{\alpha, \beta}(p) = \langle \alpha.a,\ b + \beta.(1 - \alpha.a - b)\rangle,$$

$$J_{\alpha, \beta}(p) = \langle a + \alpha.(1 - a - b),\ \beta.b\rangle,$$

$$J^{*}_{\alpha, \beta}(p) = \langle a + \alpha.(1 - a - \beta.b),\ \beta.b\rangle,$$

$$X_{\alpha, \beta, \Gamma, \partial, \varphi, \psi}(p) = \langle \alpha.a + \beta.(1 - a - \Gamma.b),\ \partial.b + \varphi.(1 - \psi.a - b)\rangle,$$

for $\alpha + \varphi - \varphi.\psi \le 1$ and $\beta + \partial - \beta.\Gamma \le 1$.

Obviously,

$$D_{\alpha}(p) = F_{\alpha, 1-\alpha}(p),$$

$\Box p = X_{1, 0, r, 1, 1, 1}(p)$,

$\Diamond p = X_{1, 1, 1, 1, 0, r}(p)$,

$D_{\alpha}(p) = X_{1, \alpha, 1, 1, 1-\alpha, 1}(p)$,

$F_{\alpha, \beta}(p) = X_{1, \alpha, 1, 1, \beta, 1}(p)$, for $\alpha + \beta \leq 1$,

$G_{\alpha, \beta}(p) = X_{\alpha, 0, r, \beta, 0, r}(p)$,

$H_{\alpha, \beta}(p) = X_{\alpha, 0, r, 1, \beta, 1}(p)$,

$H^{*}_{\alpha, \beta}(p) = X_{\alpha, 0, r, \beta, 0, \alpha}(p)$,

$J_{\alpha, \beta}(p) = X_{1, \alpha, 1, \beta, 0, r}(p)$,

$J^{*}_{\alpha, \beta}(p) = X_{1, \alpha, \beta, \beta, 0, r}(p)$,

where r is an arbitrary real number in [0, 1].

The validity of the following assertions is checked from the above definitions.

THEOREM 1: If p is a tautology, then:

(a) for every two $\alpha, \beta \in [0, 1]$ $D_{\alpha}(p)$, $F_{\alpha, \beta}(p)$ (for $\alpha + \beta \leq 1$), $J_{\alpha, \beta}(p)$ and $J^{*}_{\alpha, \beta}(p)$ are tautologies.

(b) $H_{\alpha, \beta}(p)$ and $H^{*}_{\alpha, \beta}(p)$ are IFTs.

THEOREM 2: If p is an IFT, then:

(a) for $\alpha \geq 1/2$ $D_{\alpha}(p)$ is an IFT,

(b) for $\alpha \geq \beta$ and $\alpha + \beta \leq 1$ $F_{\alpha, \beta}(p)$ is an IFT,

(c) for $\alpha \geq \beta$ $G_{\alpha, \beta}(p)$, $J_{\alpha, \beta}(p)$, $J^{*}_{\alpha, \beta}(p)$ are IFTs,

(d) for $\alpha \geq \partial$, $\beta \geq \varphi$, $\psi \geq \Gamma$ $X_{\alpha, \beta, \Gamma, \partial, \varphi, \psi}(p)$ is an IFT.

THEOREM 3: For every $a, b, \alpha, \beta, \Gamma, \partial, \varphi, \psi \in [0, 1]$ and $a + b \leq 1$, if $A \in \{D_{\alpha}, F_{\alpha, \beta}, G_{\alpha, \beta}, H_{\alpha, \beta}, H^{*}_{\alpha, \beta}, J_{\alpha, \beta}, J^{*}_{\alpha, \beta}, X_{\alpha, \beta, \Gamma, \partial, \varphi, \psi}\}$, then $A(\langle a, b\rangle) \acute{\supset} \langle a, b\rangle$ is an IFS for the (max-min)-variant of the operation implication.

It can easily be seen, that this assertion is not valid for the other type of this operation. For example,

$$F_{\alpha, \beta}(\langle a, b\rangle) \supset \langle a, b\rangle$$
$$= \langle a + \alpha.(1 - a - b), b + \beta.(1 - a - b)\rangle \supset \langle a, b\rangle$$
$$= \langle 1 - (1 - a).sg(\alpha.(1 - a - b)), b.sg(\alpha.(1 - a - b)).0\rangle$$
$$= \begin{cases} \langle 1, 0\rangle, & \text{if } \alpha = 0 \text{ or } a + b = 1 \\ \langle a, 0\rangle, & \text{if } \alpha > 0 \text{ and } a + b < 1 \end{cases}$$

THEOREM 4: For every $a, b, \alpha, \beta, \Gamma, \partial, \varphi, \psi, \alpha', \beta', \Gamma', \partial', \varphi', \psi' \in [0, 1]$ and $a + b \leq 1$, if $A \in \{D, F, G, H, H^*, J, J^*\}$, then for sg-variant of the implication:

(a) $A_{\alpha,\beta}(\langle a, b\rangle) \supset A_{\alpha',\beta'}(\langle a, b\rangle)$ is a tautology for $\alpha \leq \alpha'$,

(b) $X_{\alpha,\beta,\Gamma,\partial,\varphi,\psi}(\langle a, b\rangle) \supset X_{\alpha',\beta',\Gamma',\partial',\varphi',\psi'}(\langle a, b\rangle)$ is a tautology for $\alpha \leq \alpha'$, $\beta \leq \beta'$ and $\Gamma \geq \Gamma'$.

THEOREM 5: For every $a, b, \alpha, \beta, \Gamma, \partial, \varphi, \psi, \alpha', \beta', \Gamma', \partial', \varphi', \psi' \in [0, 1]$ and $a + b \leq 1$, if $A \in \{D, F, G, H, H^*, J, J^*\}$, then for the (max-min)-variant of the implication

(a) $A_{\alpha,\beta}(\langle a, b\rangle) \supset A_{\alpha',\beta'}(\langle a, b\rangle)$ is an IFT for $\alpha \leq \alpha'$ or $\beta \geq \beta'$,

(b) $X_{\alpha,\beta,\Gamma,\partial,\varphi,\psi}(\langle a, b\rangle) \supset X_{\alpha',\beta',\Gamma',\partial',\varphi',\psi'}(\langle a, b\rangle)$ is an IFT for $\alpha \leq \alpha'$, $\beta \leq \beta'$ and $\Gamma \geq \Gamma'$ or for $\partial \geq \partial'$, $\varphi \geq \varphi'$ and $\psi \leq \psi'$.

THEOREM 6: For every $a, b, \alpha, \beta \in [0, 1]$ and $a + b \leq 1$:

(a) $\neg F_{\alpha,\beta}(\langle a, b\rangle) = F_{\beta,\alpha}(\langle b, a\rangle)$, if $\alpha + \beta \leq 1$,

(b) $\neg G_{\alpha,\beta}(\langle a, b\rangle) = G_{\beta,\alpha}(\langle b, a\rangle)$,

(c) $\neg H_{\alpha,\beta}(\langle a, b\rangle) = J_{\beta,\alpha}(\langle b, a\rangle)$.

THEOREM 7: For every two propositions p and q and for every two $\alpha, \beta \in [0, 1]$, for the (max-min)-variant of the implication:

(a) $F_{\alpha,\beta}(p \,\&\, q) \supset F_{\alpha,\beta}(p) \,\&\, F_{\alpha,\beta}(q)$, for $\alpha + \beta \leq 1$,

(b) $F_{\alpha,\beta}(p \vee q) \subset F_{\alpha,\beta}(p) \vee F_{\alpha,\beta}(q)$, for $\alpha + \beta \leq 1$,

(c) $G_{\alpha,\beta}(p \,\&\, q) = G_{\alpha,\beta}(p) \,\&\, G_{\alpha,\beta}(q)$,

(d) $G_{\alpha,\beta}(p \vee q) = G_{\alpha,\beta}(p) \vee G_{\alpha,\beta}(q)$,

(e) $H_{\alpha,\beta}(p \,\&\, q) \supset H_{\alpha,\beta}(p) \,\&\, H_{\alpha,\beta}(q)$,

(f) $H_{\alpha,\beta}(p \vee q) \subset H_{\alpha,\beta}(p) \vee H_{\alpha,\beta}(q)$,

(g) $J_{\alpha,\beta}(p \,\&\, q) \subset H_{\alpha,\beta}(p) \,\&\, H_{\alpha,\beta}(q)$,

(h) $H_{\alpha,\beta}(p \vee q) \supset H_{\alpha,\beta}(p) \vee H_{\alpha,\beta}(q)$,

(i) $H^*_{\alpha,\beta}(p \,\&\, q) \supset H^*_{\alpha,\beta}(p) \,\&\, H^*_{\alpha,\beta}(q)$,

(j) $H^*_{\alpha,\beta}(p \vee q) \subset H^*_{\alpha,\beta}(p) \vee H^*_{\alpha,\beta}(q)$,

(k) $J^*_{\alpha,\beta}(p \,\&\, q) \subset H^*_{\alpha,\beta}(p) \,\&\, H^*_{\alpha,\beta}(q)$,

$$(1)\ H^{*}_{\alpha,\beta}(p \veebar q) \supset H^{*}_{\alpha,\beta}(p) \veebar H^{*}_{\alpha,\beta}(q).$$

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