Group Decision Making Under Fuzzy Environments

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Abstract

In this paper, we outline an approach to the fuzzy undominated solutions of group decision making under fuzzy preferences and fuzzy majority. We also investigate the fuzzy degree of consensus of group decision making. The use of fuzzy linguistic variables and the fuzzy-logic-based operations would help make group decision making models more consistent with human perception.

Keywords: group decision making, fuzzy linguistic variable, fuzzy logic, fuzzy majority, fuzzy preference.

1. Introduction

One of the most crucial human activityes is decision making. Among decision making, group decision making is an important class. Since the process of decision making, notably of group decision making, is centered on human beings with their inherent subjectivity, imprecision and vagueness in the expression of opinions, the fuzzy set theory have been used as a very natural and usful tool in this field. In this paper we try to outline an approach to the fuzzification of group decision making and consensus models.

The paper is organized as follows. In section 2, the nations of fuzzy preferences and fuzzy majority and other relevant concepts are introduced. A brief review of the fuzzy logical operations of fuzzzy linguistic variables is presented in section 3. In section 4, an approach to the undominated solutions of group decision making under fuzzy majority is outlined. Last, the section 5 is devoted to the derivation of the unanimity of group decision making under fuzzy preferences and fuzzy majority.

2. Fuzzy Preferences and Fuzzy Majority

Let $X=\{x_1, x_2, \ldots, x_n\}$ is a finite non-empty set of n alternatives to be assessed by a group of m individuals $E=\{e_1, e_2, \ldots, e_m\}$. Each individual $e_k \in E$ provides his or her preferences over the set of alternatives X. Due to the individual's inherent subjectivity, imprecision and vagueness in the articalation of opinions, the fuzzy preference relations is strongly advocated.

For individual $e_k \in E$, his or her fuzzy preference $\tilde{R}_k$ over the set of alternatives X is a fuzzy set on the direct product $X \times X$ with the membership function $\mu_{R_k}: X \times X \to [0,1]$, i.e. a binary fuzzy relation on X. The fuzzy preference $\tilde{R}_k$ may be represented by a matrix $R_k = [r_{ij}]$ and R_k is commonly assumed complementary, i.e. $r_{ij}^{(k)} + r_{ji}^{(k)} = 1$ for $1 < k < m$ and $1 < i, j < n$ with $i \neq j$ and all diagonal elements $r_{ii}^{(k)} = 0$ for $1 < k < m$ and $1 < i < n$.

Very often, one may admit that the various individuals who give the opinions are not equally important. Here we apply a fuzzy linguistic variable of "important" to the evalution of individuals' weights. The fuzzy linguistic variable of "important", $\tilde{W}$, is defined as a fuzzy set on the set of individuals $E = \{e_1, e_2, \dots, e_m\}$, written as

$$\tilde{W} = w_1 / e_1 + w_2 / e_2 + \dots + w_m / e_m \tag{2.1}$$

Where $w_k = \mu_W(e_k) \in [0, 1]$ is the degree of importance of e_k, from 1 for definitely important to 0 for definitely unimportant through all intermediate values.

Now, we consider the majority. Conventionally, the strict majority rule with $100\alpha\%$ agreement ($0.5 < \alpha < 1$, e.g., "at least half", "more then 70%", etc., is employed in group decision making. But often the required majority is imprecisely specified and it is just a soft majority or fuzzy majority given by a fuzzy linguistic variable, e.g., "most", "almost all", etc. The fuzzy majorty, $\tilde{M}$, can be assumed to be a fuzzy set defined on [0, 1]. For instance, $\tilde{M}$ = "most" may be given as

$$\mu_M(x) = \begin{cases} 1, & if \quad \frac{2}{3} \leqslant x \leqslant 1 \\ 3x - 1, & if \quad \frac{1}{3} < x < \frac{2}{3} \\ 0, & if \quad 0 \leqslant x \leqslant \frac{1}{3} \end{cases} \tag{2.2}$$

which may be interpreted as follows: if at least two thirds of the individuals approve a alternative, then most of them certainly approve it (approve to degree 1), when less than one third of the individuals approve it, then most of them certainly do not approve it (approve to degree 0), and between one third and two thirds, the more of individual approve it, the higher the degree of approval by most of them.

3. Fuzzy–Logic–Based Operation of Fuzzy Linguistic variables

In group decision making, some propositions quantified by fuzzy linguisitic variables, such as, "most individuals approve the alternative x" and "almost all important individuals approve the alternative x", are used often. Generally, a proposition quantified by fuzzy linguistic variables may be written as

$$\tilde{M} - e's \ are \ P \qquad or \qquad \tilde{M} - \tilde{W} - e's \ are \ \tilde{M} \tag{3.1}$$

where $\tilde{M}$ is a fuzzy majority, $\tilde{W}$ is a fuzzy linguistic variable of "important", $\tilde{P}$ is a property which may possesses some vagueness and therefore it can be defined as a fuzzy set on the set of individuals. For instance, $\tilde{P}$ = "approve alternative x" may be exemplified by

$$\tilde{P} = p_1 / e_1 + p_2 / e_2 + \dots + p_m / e_m \tag{3.2}$$

Which means that individual e_k approve alternative x to degree p_k $(k = 1,2,\dots m)$.

For our purposes, the p_k can be view as the truth of the propostion of "e_k is $\tilde{P}$". Then, according to Zadeh's approach [10],

$$The\ truth\ oj\ "\tilde{M} - e's\ are\ \tilde{P}" = \mu_M(y) \tag{3.3}$$

Where

$$y = \frac{1}{m}\sum_{k=1}^{m} p_k \tag{3.4}$$

is the extent to which all the individuals possesses the property $\tilde{P}$. Also we have

$$The\ truth\ oj\ "\tilde{M} - \tilde{W} - e's\ are\ \tilde{P}" = \mu_M(\tilde{y}) \tag{3.5}$$

Where

$$\tilde{y} = \frac{\sum_{k=1}^{m} p_k \wedge \mu_W(e_k)}{\sum_{k=1}^{m} \mu_W(e_k)} \tag{3.6}$$

is the extent to which all the important individuals possesses the property $\tilde{P}$.

4. Group Decision Making under Fuzzy preferences and Fuzzy Majority

Let $R_k = [\ r_{ij}^{(k)}]$be the fuzzy preference of individual $e_k \in E$over the set of alternatives $X = \{x_1, x_2, \dots x_n\}$. For a stroct majority $m_0 (m / 2 < m_0 < m)$ the set of undominated solutions U is defined as

$$U = \{x_j \in X:\ there\ is\ no\ x_i \in X\ such\ that\ r_{ij}^{(k)} > 0.5\ holds\ jor\ at\ least\ m_0\ k's\}$$

i.e. a set of alternatives not defeated in pairwisw comparisonns by the required strict majority m_0.

Suppose now that the required majority is a fuzzy majority $\tilde{M}$like "most" defined by (2.4). Let

$$p_{ij}^{(k)} = \begin{cases} 1, & ij\ \ r_{ij}^{(k)} > \frac{1}{2} \\ 0, & otherwise \end{cases} \tag{4.1}$$

for $1 < i, j < n$ and $1 < k < m$. Obviously, $p_{ij}^{(k)}$ reflects if alternative x_i defeats x_j or not according to e_k's opinion. Thus

$$p_i^{(k)} = \frac{1}{n-1}\sum_{j=1, j\neq i}^{n} p_{ij}^{(k)} \tag{4.2}$$

is the extent towhich individual $e_k$approve alternative x_i. Then the property of "approve alternative x_i" is the fuzzy set on the set of individuals E defined as

$$\tilde{P}_i = p_i^{(1)} / e_1 + p_i^{(2)} / e_2 + \ \dots \ + p_i^{(m)} / e_m \tag{4.3}$$

Notice that

$$p_i = \frac{1}{m} \sum_{k=1}^{m} p_i^{(k)} \tag{4.4}$$

is just the extent to which all the individuals approve x_i, and

$$\textit{The truth of "}\tilde{M} - \textit{individuals approve } x_i\textit{"} \quad a_M^{(i)} = \mu_M(p_i) \tag{4.5}$$

is to what extent the $\tilde{M}$-individuals approve x_i. Thus the fuzzy $\tilde{M}$-undominated solution of the group decision making is now defined as a fuzzy set on the set of alternatives X.

$$S_M = a_M^{(1)} / x_1 + a_M^{(2)} / x_2 + \ \dots \ + a_M^{(n)} / x_n \tag{4.6}$$

Analogously, by introducing a threshold α on the degree of approval in (4.1), we can define the fuzzy $\tilde{M}-\alpha-$undominated solution of the group decision making. First, we denote

$$p_{ij}^{(k)}(\alpha) = \begin{cases} 1, & \textit{if} \ \ r_{ij}^{(k)} > \alpha \geqslant \frac{1}{2} \\ 0, & \textit{otherwise} \end{cases} \tag{4.7}$$

and then, following the line of reasoning (4.2)–(4.6), we get

$$S_M(\alpha) = a_M^{(1)}(\alpha) / x_1 + a_M^{(2)}(\alpha) / x_2 + \ \dots \ + a_M^{(n)}(\alpha) / x_n \tag{4.8}$$

i.e., a fuzzy set of alternatives that are sufficiently approved (at least to degree α) by $\tilde{M}$-individuals.

5. Fuzzy Unanimity under Fuzzy Preferences and Fuzzy Majority

In this section we will employ the fuzzy linguistic variables to define a fuzzy unanimity. This unanimity is meant to overcome some rigidity of the conventional concept of consensus in which full consensus occurs only when "all the individuals agree on all the issues" may often not consistent with real human perception of the very essence of consensus.

We start with the degree of strict agreement between individuals e_s and e_t concerning their preferences between alternatives x_i and x_j $(i \neq j)$

$$d_{ij}(e_s, e_t) = \begin{cases} 1, & \textit{if} \ \ r_{ij}^{(s)} = r_{ij}^{(t)} \\ 0, & \textit{otherwise} \end{cases} \tag{5.1}$$

for $1 < s, t < m$ and $1 < i, j < n$. The degree of agreement between individuals e_s and e_t concerning their preferences between all the pairs of alternatives is

$$d(e_s, e_t) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij}(e_s, e_t)}{n^2} \tag{5.2}$$

The degree of agreement between e_s and e_t concerning their preferences between $\tilde{M}_1$ −pairs of alternatives is

$$r_{M_1}(e_s, e_t) = \mu_{M_1}(d(e_s, e_t)) \tag{5.3}$$

For the non−homogeneous group of individuals, the importance of individuals is a fuzzy set $\tilde{W}$ on E given by (2.1). Then the importance of pairs of individuals, $\tilde{Q}_1$, can be viewed as a fuzzy set on the direct product E × E, and it may be defined in various ways among which

$$\tilde{Q}_1 = \sum_{s=1}^{m}\sum_{t=1}^{m}\frac{1}{2}(\mu_W(e_s) + \mu_W(e_t)) / (e_s, e_t) \tag{5.4}$$

is certainly the most straightforward.

In turn, the degree of agreement of all the pairs of important individuals concerning their preference between $\tilde{M}_1$ −pairs of alternatives is

$$r_{M_1}(\tilde{M}) = \frac{\sum_{s=1}^{m}\sum_{t=1}^{m} r_{M_1}(e_s, e_t) \wedge \frac{1}{2}(\mu_W(e_s) + \mu_W(e_t))}{\sum_{s=1}^{m}\sum_{t=1}^{m}\frac{1}{2}(\mu_W(e_s) + \mu_W(e_t))} \tag{5.5}$$

Finally, the degree of agreement of $\tilde{M}_2$ −pairs of important individuals concerning their preferences between $\tilde{M}_1$ −pairs of alternatives, called the $(\tilde{M}_1, \tilde{M}_2, \tilde{W})$−fuzzy unanimity, is

$$U(\tilde{M}_1, \tilde{M}_2, \tilde{W}) = \mu_{M_2}(r_{M^1}(\tilde{W})) \tag{5.6}$$

Analogously, by introducing a threshold α $(1/2 < \alpha < 1)$ on the degree of strict agreement (5.1), we can define the α−degree of agreement of individuals e_s and e_t concerning their preferences between alternatives x_i and x_j by

$$d_{ij}(\alpha)(e_s, e_t) = \begin{cases} 1, & if \ |r_{ij}(s) - r_{ij}(t)| \leq 1 - \alpha \\ 0, & otherwise \end{cases} \tag{5.7}$$

Then,following the reasoning (5.2)−(5.6), we obtain the α−degree of agreement of $\tilde{M}_2$ −pairs individuals concerning their preferences between $\tilde{M}_1$ −pairs of alternatives, called the α $(\tilde{M}_1, \tilde{M}_2, \tilde{W})$−fuzzy unanimity, given by

$$U_\alpha(\tilde{M}_1, \tilde{M}_2, \tilde{W})) = \mu_{M_2}(r_{M_1}(\tilde{W}, \alpha)) \tag{5.8}$$

Where

$$r_{M_1}(\tilde{W}, \alpha) = \frac{\sum_{s=1}^{m}\sum_{t=1}^{m} r_{M_1}(\alpha)(e_s, e_t) \wedge \frac{1}{2}(\mu_W(e_s) + \mu_W(e_t))}{\sum_{s=1}^{m}\sum_{t=1}^{m}\frac{1}{2}(\mu_W(e_s) + \mu_W(e_t))} \tag{5.9}$$

$$r_{M_1}(\alpha)(e_s, e_t) = \mu_{M_1}(d(\alpha)(e_s, e_t)) \tag{5.10}$$

$$d(\alpha)(e_s, e_t) = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}(\alpha)(e_s, e_t) \tag{5.11}$$

6. Conclusions

In this paper we have tried to outline an approach to the fuzzification of group decision making and consensus models. The fuzzy–logic–based operations of fuzzy linguistic variables have provided a usful tool. The fuzzy undominated solutions and fuzzy unanimity of the group decision making have been devised.

As the restriction of the length extent, we omit the number examples for the approaches outlined in this paper.

References

[1] Dubois D. and Prade H., *Fuzzy Sets and Systems: Theory and Applications* (Academic Press, New York, 1980).

[2] Fedrizzi M., Kacprzyk J. and Zadrozny S., An interactive multi–user decision support system for consensus reaching processes using fuzzy logic with linguistic quantifiers, *Decision Support Systems* 4 (1988) 313–327.

[3] Kacprzyk J., Group decision making with a fuzzylinguistic majority, *Fuzzy Sets and Systems* 18 (1986) 110–118.

[4] Kacprzyk J. and Fedrizzi M., 'Soft' consensus measures for monitioring real consensous reaching processes under fuzzy preferences, *Control Cybernet.* 15 (1986) 309–323.

[5] Kacprzyk J. and Fedrizzi M., A 'soft' measure of consensous in the setting of partical (fuzzy) preferences, *European J. Oper. Res.* 34 (1988) 315–325.

[6] Kacprzyk J. and Roubens M., Eds., *Non–conventional Preference Relations in Decision Making* (Springer–dam, 1989) 267–281.

[7] Kacprzyk J. and Fedrizzi M., Eds., *Multiperson Decision Making Models Using Fuzzy Sets and Possibility Theory* (Kluwer, Dordrecht, 1990).

[8] Montero F.J., Aggregation of fuzzy opinions in a non–homogeneous group, *Fuzzy Sets and Systems* 25 (1987) 15–20.

[9] Nurmi H., Approaches to collective decision making with fuzzy preference relations, *Fuzzy Sets and Systems* 6 (1981) 249–259.

[10] Zadeh L.A., A computationalapproach to fuzzy quantifiers in natyral languages, *Comput. Math. Appl.* 9 (1983) 149–184.