NEW OPERATION, DEFINED OVER THE INTUITIONISTIC FUZZY SETS. 3

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Some operations (as "$\cup$", "$\cap$", "+", ".", "@", "$") are defined over the Intuitionstic Fuzzy Sets (IFSs) in [1-4]. Here we shall introduce a new one, and we shall show its basic properties.

Let a set E be fixed. An IFS A^* in E is an object having the form:

$$A^* = \{<x, \mu_A(x), \gamma_A(x)>/x \in E\},$$

where the functions $\mu_A : E \to [0, 1]$ and $\gamma_A : E \to [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in E$ to the set A, which is a subset of E, respectively, and for every $x \in E$:

$$0 \le \mu_A(x) + \gamma_A(x) \le 1.$$

For every two IFSs A and B are valid (see [1-4]) the following definitions (let $\alpha, \beta \in [0, 1]$):

$$A \subset B \text{ iff } (\forall x \in E)(\mu_A(x) \le \mu_B(x) \;\&\; \gamma_A(x) \ge \gamma_B(x));$$

$$A \supset B \text{ iff } B \subset A;$$

$$A = B \text{ iff } (\forall x \in E)(\mu_A(x) = \mu_B(x) \;\&\; \gamma_A(x) = \gamma_B(x))$$

$$A \subset_{\Box} B \text{ iff } (\forall x \in E)(\mu_A(x) \le \mu_B(x));$$

$$A \subset_{\Diamond} B \text{ iff } (\forall x \in E)(\gamma_A(x) \ge \gamma_B(x));$$

$$\bar{A} = \{<x, \gamma_A(x), \mu_A(x)>/x \in E\};$$

$$A \cap B = \{<x, \min(\mu_A(x), \mu_B(x)), \max(\gamma_A(x), \gamma_B(x))>/x \in E\};$$

$$A \cup B = \{<x, \max(\mu_A(x), \mu_B(x)), \min(\gamma_A(x), \gamma_B(x))>/x \in E\};$$

$$A + B = \{<x, \mu_A(x)+\mu_B(x)-\mu_A(x).\mu_B(x), \gamma_A(x).\gamma_B(x)>/x \in E\};$$

$$A \;.\; B = \{<x, \mu_A(x).\mu_B(x), \gamma_A(x)+\gamma_B(x)-\gamma_A(x).\gamma_B(x)>/x \in E\};$$

$$A \;@\; B = \{<x, (\mu_A(x)+\mu_B(x))/2, (\gamma_A(x)+\gamma_B(x))/2>/x \in E\};$$

$$A \;\$\; B = \{<x, \sqrt{\mu_A(x).\mu_B(x)}, \sqrt{\gamma_A(x).\gamma_B(x)}>/x \in E\};$$

$$\Box A = \{<x, \mu_A(x), 1-\mu_A(x)>/x \in E\};$$

$\Diamond A = \{<x, 1-\gamma_A(x), \gamma_A(x)>/x \in E\}$.

Here we shall define the new operation:

$$A \# B = \{<x, 2.\mu_A(x).\mu_B(x)/(\mu_A(x) + \mu_B(x)), 2.\gamma_A(x).\gamma_B(x)/(\gamma_A(x) + \gamma_B(x))>/x \in E\}$$

for which we shall accept that if $\mu_A(x) = \mu_B(x) = 0$, then $\mu_A(x).\mu_B(x)/(\mu_A(x) + \mu_B(x)) = 0$ and if $\gamma_A(x) = \gamma_B(x) = 0$, then $\gamma_A(x)\gamma_B(x)/(\gamma_A(x) + \gamma_B(x)) = 0$.

THEOREM 1: For every two IFSs A and B, A # B is an IFS.

Proof: We must prove, that for every $x \in E$:

$$2.(\mu_A(x).\mu_B(x)/(\mu_A(x) + \mu_B(x)) + \gamma_A(x).\gamma_B(x)/(\gamma_A(x) + \gamma_B(x))) \leq 1.$$

We shall use the following

LEMMA: If $a, b, c, d \in [0, 1]$ and $a + b \leq 1$, $c + d \leq 1$, $a + c > 0$, $b + d > 0$, then:

$$\frac{2.a.c}{a + c} + \frac{2.b.d}{b + d} \leq 1. \tag{1}$$

Obviously,

$$(a.b.c + a.b.d) + (a.c.d + b.c.d) \leq a.b + c.d. \tag{2}$$

Let $a \geq c$. If $a.d \geq b.c$, then

$a.b.c + a.b.d + a.c.d + b.c.d = a.d.(b + c) + b.c.(a + d)$

$\leq a.d.(1 - a + c) + b.c.(a + 1 - c)$

$= a.d + b.c - (a.d - b.c).(a - c) \leq a.d + b.c$,

i.e.

$$a.b.c + a.b.d + a.c.d + b.c.d \leq a.d + b.c. \tag{3}$$

Let $a \geq c$. If $a.d < b.c$, then obviously, $b > d$ and

$a.b.c + a.b.d + a.c.d + b.c.d = a.d.(b + c) + b.c.(a + d)$

$\leq a.d.(b + 1 - d) + b.c.(1 - b + d)$

$= a.d + b.c - (b.c - a.d).(b - d) \leq a.d + b.c$,

i.e. (3) is valid, too.

When $a < c$ the validity of (3) is checked analogically. Adding (2) and (3) we obtain (1), from where the validity of the Lemma follows.

From the above definitions follows the validity of

THEOREM 2: For every three IFSs A, B and C:

(a) $A \# B = B \# A$;

(b) $\overline{\bar{A} \# \bar{B}} = A \# B$;

(c) $(A \cap B) \# C = (A \# C) \cap (B \# C)$;

(d) $(A \cup B) \# C = (A \# C) \cup (B \# C)$;

The other relations between the operation "#" and the other operations are not valid.

THEOREM 3: For every two IFSs A and B and for every two numbers $\alpha, \beta \in [0, 1]$:

(a) $\Box(A \# B) \supset \Box A \# \Box B$,

(b) $\Diamond(A \# B) \subset \Diamond A \# \Diamond B$,

(c) $G_{\alpha,\beta}(A \# B) = G_{\alpha,\beta}(A) \# G_{\alpha,\beta}(B)$,

Easily can be seen, that the equality:

$$(A \# B) \# C = A \# (B \# C)$$

is not valid.

THEOREM 4 (cf. [4]): For every two IFSs A and B:

(a) $A \,.\, B \subset A \cap B \subset \left\{ \begin{matrix} A \# B \\ A \$ B \\ A @ B \end{matrix} \right\} \subset A \cup B \subset A + B$;

(b) $A \# B \subset_{\Box} A \$ B \subset_{\Box} A @ B$;

(c) $A @ B \subset_{\Diamond} A \$ B \subset_{\Diamond} A \# B$.

The validities of these inclusions follow from the validity of the following inequalities for the arbitrary real numbers $a, b \in [0, 1]$:

$$a.b \le \min(a, b) \le \frac{2.a.b}{a + b} \le \sqrt{a.b} \le \frac{a + b}{2} \le \max(a, b) \le a + b - a.b.$$

This operation can be transferred to an operation on ordinary fuzzy sets as follows:

$$A \# B = \{<x, 2.\mu_A(x).\mu_B(x)/(\mu_A(x) + \mu_B(x))>/x\in E\}.$$

Now, Theorem 4 obtains the form:

$$A \,.\, B \subset A \cap B \subset A \# B \subset A \$ B \subset A @ B \subset A \cup B \subset A + B.$$

REFERENCES:

[1] Atanassov K. Intuitionistic fuzzy sets, Fuzzy sets and Systems Vol. 20 (1986), No. 1, 87-96.

[2] Atanassov, K. More on intuitionistic fuzzy sets. Fuzzy sets and systems, 33, 1989, No. 1, 37-46.

[3] Atanassov, K. New operation, defined over the intuitionistic fuzzy sets. submitted to BUSEFAL.

[4] Atanassov, K. New operation, defined over the intuitionistic fuzzy sets. 2., submitted to BUSEFAL.