NEW OPERATION, DEFINED OVER THE INTUITIONISTIC FUZZY SETS

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Some operations (as "U", "∩", "+", ".") are defined over the Intuitionstic Fuzzy Sets (IFSs) in [1]. Here we shall introduce a new one, and we shall show its basic properties.

Let a set E be fixed. An IFS A^* in E is an object having the form:

$$A^* = \{<x, \mu_A(x), \gamma_A(x)>/x \in E\},$$

where the functions $\mu_A : E \to [0, 1]$ and $\gamma_A : E \to [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in E$ to the set A, which is a subset of E, respectively, and for every $x \in E$:

$$0 \le \mu_A(x) + \gamma_A(x) \le 1.$$

Obviously, every ordinary fuzzy set has the form:

$$\{<x, \mu_A(x), 1-\mu_A(x)>/x \in E\}.$$

For every two IFSs A and B are valid (see [1-4]) the following definitions (let $\alpha, \beta \in [0, 1]$):

$A \subset B$ iff $(\forall x \in E)(\mu_A(x) \le \mu_B(x)\ \&\ \gamma_A(x) \ge \gamma_B(x))$;

$A \supset B$ iff $B \subset A$;

$A = B$ iff $(\forall x \in E)(\mu_A(x) = \mu_B(x)\ \&\ \gamma_A(x) = \gamma_B(x))$

$\bar{A} = \{<x, \gamma_A(x), \mu_A(x)>/x \in E\}$;

$A \cap B = \{<x, \min(\mu_A(x), \mu_B(x)), \max(\gamma_A(x), \gamma_B(x))>/x \in E\}$;

$A \cup B = \{<x, \max(\mu_A(x), \mu_B(x)), \min(\gamma_A(x), \gamma_B(x))>/x \in E\}$;

$A + B = \{<x, \mu_A(x)+\mu_B(x)-\mu_A(x).\mu_B(x), \gamma_A(x).\gamma_B(x)>/x \in E\}$;

$A . B = \{<x, \mu_A(x).\mu_B(x), \gamma_A(x)+\gamma_B(x)-\gamma_A(x).\gamma_B(x)>/x \in E\}$;

$\Box A = \{<x, \mu_A(x), 1-\mu_A(x)>/x \in E\}$;

$\Diamond A = \{<x, 1-\gamma_A(x), \gamma_A(x)>/x \in E\}$;

$C(A) = \{<x, K, L>/x \in E\}$,

where $K = \max_{x \in E} \mu_A(x)$, $L = \min_{x \in E} \gamma_A(x)$;

$I(A) = \{<x, K, l>/x \in E\}$,

where $k = \min_{x \in E} \mu_A(x)$, $l = \max_{x \in E} \gamma_A(x)$;

$D_\alpha(A) = \{<x, \mu_A(x)+\alpha.\pi_A(x), \gamma_A(x)+(1-\alpha).\pi_A(x)>/x \in E\}$;

$F_{\alpha,\beta}(A) = \{<x, \mu_A(x)+\alpha.\pi_A(x), \gamma_A(x)+\beta.\pi_A(x)>/x \in E\}$,

where $\alpha + \beta \leq 1$;

$G_{\alpha,\beta}(A) = \{<x, \alpha.\mu_A(x), \beta.\gamma_A(x)>/x \in E\}$;

$H_{\alpha,\beta}(A) = \{<x, \alpha.\mu_A(x), \gamma_A(x)+\beta.\pi_A(x)>/x \in E\}$;

$H^*_{\alpha,\beta}(A) = \{<x, \alpha.\mu_A(x), \gamma_A(x)+\beta.(1-\alpha.\mu_A(x)-\gamma_A(x))>/x \in E\}$;

$J_{\alpha,\beta}(A) = \{<x, \mu_A(x)+\alpha.\pi_A(x), \beta.\gamma_A(x)>/x \in E\}$;

$J^*_{\alpha,\beta}(A) = \{<x, \mu_A(x)+\alpha.(1-\mu_A(x)-\beta.\gamma_A(x)), \beta.\gamma_A(x)>/x \in E\}$.

Now, we shall define the new operation (cf. [5]):

$$A @ B = \{<x, (\mu_A(x)+\mu_B(x))/2, (\gamma_A(x)+\gamma_B(x))/2>/x \in E\}.$$

Obviously, the set $A @ B$ is an IFS.

THEOREM 1: For every three IFSs A, B and C:

(a) $A @ B = B @ A$;

(b) $\overline{\bar{A} @ \bar{B}} = A @ B$;

(c) $(A \cap B) @ C = (A @ C) \cap (B @ C)$;

(d) $(A \cup B) @ C = (A @ C) \cup (B @ C)$;

(e) $(A + B) @ C \subset (A @ C) + (B @ C)$;

(f) $(A . B) @ C \supset (A @ C) . (B @ C)$;

(g) $(A @ B) + C = (A + C) @ (B + C)$;

(h) $(A @ B) . C = (A . C) @ (B . C)$.

Proof: (e) Initially we shall prove, that for every three numbers $a, b, c \in [0, 1]$ it is valid the inequality:

$$c.(2 - a - b - c) + a.b \geq 0. \qquad (*)$$

When $c^2 \leq a.b$, then:

$$c.(2 - a - b - c) + a.b = c.(2 - a - b) - c^2 + a.b \geq a.b - c^2 \geq 0.$$

When $c^2 > a.b$, then:

$$c.(2 - a - b - c) + a.b \geq \sqrt{a.b}.(1 - a - b + \sqrt{a.b})$$

$$\geq \begin{cases} \sqrt{a.b}.((1 - a) + b.(a - b)) \geq 0, & \text{if } a \geq b \\ \sqrt{a.b}.((1 - b) + a.(b - a)) \geq 0, & \text{if } a < b \end{cases}.$$

Let A, B and C be three given IFSs. Then:

$(A + B) @ C = \{<x, (\mu_A(x)+\mu_B(x)-\mu_A(x).\mu_B(x)+\mu_C(x))/2,$

$(\gamma_A(x).\gamma_B(x)+\gamma_C(x))/2>/ x \in E \}$

$(A @ C) + (B @ C) = \{<x, (\mu_A(x)+\mu_C(x))/2 + (\mu_B(x)+\mu_C(x))/2 -$

$(\mu_A(x)+\mu_C(x)).(\mu_B(x)+\mu_C(x))/4, (\gamma_A(x)+\gamma_C(x)).(\gamma_B(x)+\gamma_C(x))/4>$

$/ x \in E \}.$

Using (*), from:

$(\mu_A(x)+\mu_C(x))/2 + (\mu_B(x)+\mu_C(x))/2 - (\mu_A(x)+\mu_C(x)).(\mu_B(x)+\mu_C(x))/4$

$- (\mu_A(x)+\mu_B(x)-\mu_A(x).\mu_B(x)+\mu_C(x))/2$

$= (2.\mu_C(x)+\mu_A(x).\mu_B(x)-\mu_A(x).\mu_C(x)-\mu_B(x).\mu_C(x)-\mu_C(x)^2)/4 \geq 0$

and form:

$(\gamma_A(x).\gamma_B(x)+\gamma_C(x))/2 - (\gamma_A(x)+\gamma_C(x)).(\gamma_B(x)+\gamma_C(x))/4$

$= (2.\gamma_C(x)+\gamma_A(x).\gamma_B(x)-\gamma_A(x).\gamma_C(x)-\gamma_B(x).\gamma_C(x)-\gamma_C(x)^2)/4 \geq 0$

we see that (e) is valid.

The other assertions are proved analogically. The other relations between the operation "@" and the above operations are not valid. By the same means is proved the following theorem, also.

THEOREM 2: For every two IFSs A and B and for every two numbers $\alpha, \beta \in [0, 1]$:

(a) $\Box(A @ B) = \Box A @ \Box B$,

(b) $\Diamond(A @ B) = \Diamond A @ \Diamond B$,

(c) $D_\alpha(A @ B) = D_\alpha(A) @ D_\alpha(B)$,

(d) $F_{\alpha,\beta}(A @ B) = F_{\alpha,\beta}(A) @ F_{\alpha,\beta}(B)$, for $\alpha + \beta \leq 1$,

(e) $G_{\alpha,\beta}(A @ B) = G_{\alpha,\beta}(A) @ G_{\alpha,\beta}(B)$,

(f) $H_{\alpha,\beta}(A @ B) = H_{\alpha,\beta}(A) @ H_{\alpha,\beta}(B)$,

(g) $H^*_{\alpha,\beta}(A @ B) = H^*_{\alpha,\beta}(A) @ H^*_{\alpha,\beta}(B)$,

(i) $J_{\alpha,\beta}(A @ B) = J_{\alpha,\beta}(A) @ J_{\alpha,\beta}(B)$,

(j) $J^*_{\alpha,\beta}(A @ B) = J^*_{\alpha,\beta}(A) @ J^*_{\alpha,\beta}(B)$,

(k) $C(A @ B) \subset C(A) @ C(B)$,

(l) $I(A @ B) \supset I(A) @ I(B)$,

Easily can be seen, that the equality:

$$(A @ B) @ C = A @ (B @ C)$$

is not valid. Thus we define:

$$A_1 @ A_2 = \overset{2}{\underset{i=1}{@}} A_i$$

and:

$$\overset{n}{\underset{i=1}{@}} A_i = \{<x, (\sum_{i=1}^{n} \mu_{A_i}(x))/n, (\sum_{i=1}^{n} \gamma_{A_i}(x))/n> / x \in E\}.$$

Now, the following equation will be valid:

THEOREM 3: For every n+1 IFSs $A_1, A_2, \ldots, A_n$ and B:

(a) $\overline{\overset{n}{\underset{i=1}{@}} \bar{A}_i} = \overset{n}{\underset{i=1}{@}} A_i$;

(b) $\overset{n}{\underset{i=1}{@}} A_i + B = \overset{n}{\underset{i=1}{@}} (A_i + B)$;

(c) $\overset{n}{\underset{i=1}{@}} A_i \,.\, B = \overset{n}{\underset{i=1}{@}} (A_i \,.\, B)$;

THEOREM 4: For every n IFSs $A_1, A_2, \ldots, A_n$ and for every two numbers $\alpha, \beta \in [0, 1]$:

(a) $\Box(\overset{n}{\underset{i=1}{@}} A_i) = \overset{n}{\underset{i=1}{@}} \Box A_i$;

(b) $\Diamond(\overset{n}{\underset{i=1}{@}} A_i) = \overset{n}{\underset{i=1}{@}} \Diamond A_i$;

(c) $D_\alpha(\overset{n}{\underset{i=1}{@}} A_i) = \overset{n}{\underset{i=1}{@}} D_\alpha(A_i)$;

(d) $F_{\alpha,\beta}(\overset{n}{\underset{i=1}{@}} A_i) = \overset{n}{\underset{i=1}{@}} F_{\alpha,\beta}(A_i)$, for $\alpha + \beta \leq 1$;

(e) $G_{\alpha,\beta}(\overset{n}{\underset{i=1}{@}} A_i) = \overset{n}{\underset{i=1}{@}} G_{\alpha,\beta}(A_i)$;

(f) $H_{\alpha,\beta}(\overset{n}{\underset{i=1}{@}} A_i) = \overset{n}{\underset{i=1}{@}} H_{\alpha,\beta}(A_i)$;

(g) $H^*_{\alpha,\beta}(\overset{n}{\underset{i=1}{@}} A_i) = \overset{n}{\underset{i=1}{@}} H^*_{\alpha,\beta}(A_i)$;

(h) $J_{\alpha,\beta}(\overset{n}{\underset{i=1}{@}} A_i) = \overset{n}{\underset{i=1}{@}} J_{\alpha,\beta}(A_i)$;

(i) $J^*_{\alpha,\beta}(\overset{n}{\underset{i=1}{@}} A_i) = \overset{n}{\underset{i=1}{@}} J^*_{\alpha,\beta}(A_i)$;

(j) $C(\bigoplus_{i=1}^{n} A_i) \subset \bigoplus_{i=1}^{n} C(A_i)$;

(k) $I(\bigoplus_{i=1}^{n} A_i) \supset \bigoplus_{i=1}^{n} I(A_i)$.

The proofs of these assertions are as the above proof.

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