DRASTIC-LIKE TRIANGULAR NORMS

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Discontinous t-norms are examined. The smallest, called drastic, t-norm is generalized in a family of ϑ-drastic and π-drastic t-norms. Some examples of drastic-like t-norms are gived.

Key-words: drastic, ϑ-drastic, π-drastic and drasticlike t-norm.

0. CONVENTIONS.

By opportunity we write, indifferently but without ambiguity,

$T(x,y)$ = operation as

$x \, T \, y$ = operator as

$x \, y$ unsigned product;

$\wedge$ INF (MIN), $\vee$ SUP (MAX) (resp. if finitely)

$-$ arithmetical difference

$\setminus$ set-theoretical difference ($A \setminus B = \{x \in A \mid x \notin B\}$)

1. PRELIMINAIRES.

Let $I = [0,1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ be the unit real interval.
Any operation $I \times I \longrightarrow I$ is called operation in I;

Now we give a labelized list of operational conditions:

(C)	$x\,y = y\,x$	commutativity
(A)	$(xy)z = x(yz)$	associativity
(M)	$x < y,\ a \Rightarrow xa \leq ya$	monotony
(T1)	$1\,x = x$	1-left unity
(TO)	$x\,0 = 0$	
(SO)	$0\,x = x$	0-left unity
(S1)	$x\,1 = 1$	
(MM)	$x \leq y,\ a \leq b \Rightarrow xa \leq yb$	

1.1 **Lemma.**

- $M,\ T1 \Rightarrow T0$
- $M,\ S0 \Rightarrow S1$
- $M,\ C \Rightarrow MM$
- $MM \Rightarrow M.$

1.2 **Definition.**

A 2-place real function $I \times I \rightarrow I$ with the proprieties A,C,M is called:

- t-norm (shorted by <u>triangular_norm</u>) if it fulfills the T1;
- s-norm if it fulfills the S0 condition.

1.3 **Proposition** (boundary conditions).

For any t-norm T and s-norm S it is always:

- $T(x,0) = T(0,x) = T(0,0) = S(0,0) = 0$
- $T(x,1) = T(1,x) = S(x,0) = S(0,x) = x$
- $T(1,1) = S(1,x) = S(x,1) = S(1,1) = 1$
- $\{x,y\} \cap \{0,1\} \neq \emptyset \Rightarrow T(x,y) = x \wedge y,\quad S(x,y) = x \vee y$

1.4 **Examples.**

a) Gived $\vartheta \in]0,1]$ the function $T(x,y) = \begin{cases} \vartheta & \text{iff} \quad 0 \notin \{x,y\} \\ 0 & \phantom{\text{iff}} \quad 0 \in \{x,y\} \end{cases}$

is NO t-norm, because it fulfills C,A,M and T0 conditions, but not the T1 condition.

b) The functions $I \times I \rightarrow I$ s.t. $\forall (x,y) \in I^2$:

$T_D(x,y) = \begin{cases} 0 & \text{iff} \quad 1 \notin \{x,y\} \\ \{x,y\} \setminus \{1\} & \phantom{\text{iff}} \quad 1 \in \{x,y\} \end{cases}$ <u>drastic</u>

$T_B(x,y) = 0 \vee (x + y - 1)$ <u>bounded difference</u>

$T_A(x,y) = x \cdot y$ <u>algebraic product</u>

$T_L(x,y) = x \wedge y$ <u>logical</u>

are t-norms.

1.5 **Lemma.**

For any t-norm T and $(x,y) \in I^2$ it results:

$T_D(x,y) \leq T(x,y) \leq T_L(x,y)$.

1.6 **Proposition.**

Let $\mathcal{T}, \mathcal{S}$ be the totality of t-norms and s-norms, resp.

By composing the applications:

$\tau : \mathcal{T} \to \mathcal{S}$ s.t. $(\tau(T))(x,y) = 1 - T(1 - x, 1 - y)$

$\sigma : \mathcal{S} \to \mathcal{T}$ s.t. $(\sigma(S))(x,y) = 1 - S(1 - x, 1 - y)$

then $\forall T \in \mathcal{T}$ $\forall S \in \mathcal{S}$ it is:

$(\tau \circ \sigma)(T) = T$, $(\sigma \circ \tau)(S) = S$.

The s-norm $\tau(T)$ is called <u>T-conorm</u> (or dual t-norm) and $\sigma(S)$ is called <u>S-conorm</u>; any couple $(T, \tau(T))$ or $(\sigma(S), S)$ is called <u>conjugate pair</u> of t-norms.

1.7 **Examples.**

The t-conorms of T_L, T_A, T_B and T_D are gived, resp., by:

$S_L(x,y) = x \vee y$

$S_A(x,y) = x + y - xy$

$S_B(x,y) = 1 \wedge (x + y)$

$$S_D(x,y) = \begin{cases} 1 & \text{iff } 0 \notin \{x,y\} \\ \{x,y\} \setminus \{0\} & \phantom{\text{iff }} 0 \in \{x,y\} \end{cases}$$

1.8 **Lemma.**

$\forall S \in \mathcal{S}$ and $\forall (x,y) \in I^2$ it is: $S_L(x,y) \leq S(x,y) \leq S_D(x,y)$.

2. <u>COMBINED DRASTIC-LOGICAL T-NORMS.</u>

<u>ϑ-drastic t-norms.</u>

2.1 **Proposition.**

$\forall \vartheta \in I$ the functions $\bigwedge^{\vartheta}, \bigvee^{\vartheta} \in I^{I\times I}$ defined by:

$$\bigwedge^{\vartheta}(x,y) = \begin{cases} 0 & \text{iff } x,y \in [0,\vartheta[\\ x \wedge y & \phantom{\text{iff }} \text{otherwise} \end{cases} \qquad \bigvee^{\vartheta}(x,y) = \begin{cases} 1 & \text{iff } x,y \in]1-\vartheta, 1] \\ x \vee y & \phantom{\text{iff }} \text{otherwise} \end{cases}$$

and $\forall \vartheta \in [0,1[$ the functions $\Lambda_\vartheta, V_\vartheta$ defined by:

$$\Lambda_\vartheta(x,y) = \begin{cases} \vartheta & \text{iff } x,y \in [\vartheta,1[\\ x \wedge y & \text{otherwise} \end{cases} \qquad V_\vartheta(x,y) = \begin{cases} 1-\vartheta & \text{iff } x,y \in [0,\ 1-\vartheta] \\ x \vee y & \text{otherwise} \end{cases}$$

determine a conjugate pair of t-norms.

2.2 <u>Remark.</u>

$\Lambda_0 = T_D = \Lambda^1, \quad \Lambda^0 = T_L.$

<u>π-drastic t-norms.</u>

Let $\pi = \{P_j\}_{j \in J}$ be a partition of I such that

$\forall j \in J \;]\wedge P_j, \vee P_j[\subseteq P_j \subseteq [\wedge P_j, \vee P_j]$.

If $x \in P_j$ we write $[x]_\pi = P_j$ (equivalence classe).

Let $\mu_\pi : I \to I$ s.t. $\mu_\pi(x) = \begin{cases} 0 & x = 0 \\ 1 \text{ iff} & x = 1 \\ \wedge [x]_\pi & \text{otherwise} \end{cases}$

2.3 <u>Proposition.</u>

By writing $\mu_\pi(x) = \bar{x}$, the function $T_\pi : I \times I \to I$ s.t.

$$x \, T_\pi \, y = \begin{cases} 0 & \text{iff } 0 \in \{x,y\} \\ \underline{x} \wedge \underline{y} & 1 \in \{x,y\} \\ \bar{x} \wedge \bar{y} & \text{otherwise} \end{cases}$$

is a, called <u>π-drastic</u>, t-norm.

2.4 <u>Corollary.</u>

a) $[0,\vartheta[$ and the totality of all singletons $\{x\}$ with $x \geqslant \vartheta$ determine a partition π^ϑ of I:

Λ^ϑ is the its π^ϑ-drastic t-norm;

b) $[\vartheta,1[$, $\{1\}$ and the totality of all singletons $\{x\}$ with $x < \vartheta$ determine a partition π_ϑ of I:

Λ_ϑ is the its π_ϑ -drstic t-norm.

<u>Drastic-like t-norms.</u>

The drastic-like t-norm

$$x \Delta_\vartheta y = \begin{cases} 0 & x,y \in [0,\vartheta[\\ \vartheta & \text{iff } x,y \in [\vartheta,\ 1[\\ x \wedge y & \text{otherwise} \end{cases}$$

is not a π-drastic t-norm and it is such that:

$$x \Delta_{\vartheta} y \leq x \wedge^{\vartheta} y \quad \text{and} \quad x \Delta_{\vartheta} y \leq x \wedge_{\vartheta} y \quad .$$

Conclusions.

The drastic-like t-norms can be usefull in defuzzification problems.

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