The Entropy of Finite Product g_λ-measure Space

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Abstract: In this paper, we introduce the concept of g_λ- measure and its entropy on a finite product space. Some properties of this g_λ-measure and its entropy are invesigated.

Keywords: product space, g_λ-measure, Shannon entropy.

1. Introduction

In the classical probability theory, probability measure and its shannon entropy are studied. The concept of g_λ-measures on measurable spaces was first introduced by Sugeno[1], and it has been studied extensively by many authors (see, e.g.[2]-[7]). In this paper, we deal with the g_λ-measure and its shannon entropy on a finite product space. Some properties of this g_λ-measure and its entropy are investigated. These properties are similar to those obtained in the classical probability theory.

2. The g_λ-measure on a Finite Product Space

Throughout this paper, let $X=\{x_1,x_2...x_n\}$, $Y=\{y_1,y_2...y_m\}$ be two finite sets, $X\times Y=\{(x_i,y_j)|\ x_i\subset X,\ y_j\subset Y\}$ be the finite product space, and $P(X\times Y)$ the power set of $X\times Y$. We assume the parameter $\lambda\in(-1,\ 0)\cup(0,\ \infty)$.

Definition 2.1 A set function $g_\lambda: P(X\times Y)\rightarrow[0,\ 1]$ is called a g_λ-measure on $(X\times Y,\ P(X\times Y))$ iff it satisfies the following conditions:

(1) $0<g_{ij}<1$ $(1<i<n,\ 1<j<m)$ and

$$\prod_{i=1}^{n}\prod_{j=1}^{m}(1+\lambda g_{ij}) = 1+\lambda \quad (2.1)$$

(2) $$g_\lambda(E)= \frac{1}{\lambda}\cdot\Big[\prod_{(xi,yj)\in E}(1+\lambda g_{ij}) - 1\Big]\ \ \forall\ E\in P(X\times Y) \quad (2.2)$$

Where $$g_{ij}=g_\lambda(\{x_i,y_j\}))\ (1<i<n,\ 1<j<m) \quad (2.3)$$

The triplet $(X\times Y,\ P(X\times Y),\ g_\lambda)$ is called product g_λ-measure space.

For a product g_λ-measure space $(X\times Y,\ P(X\times Y),\ g_\lambda)$, let

$$g_{i.}=\frac{1}{\lambda}\cdot\Big[(1+\lambda)^{\sum_{j=1}^{m}\log_{1+\lambda}(1+\lambda g_{ij})} - 1\Big]\ \ (1<i<n) \quad (2.4)$$

$$g_{.j}=\frac{1}{\lambda}\cdot\left[(1+\lambda)^{\sum_{i=1}^{n}\log_{1+\lambda}(1+\lambda g_{ij})}-1\right]\quad (1\leq j\leq m) \tag{2.5}$$

Obviously, we have

$$\log_{1+\lambda}(1+\lambda g_{i.})=\sum_{j=1}^{m}\log_{1+\lambda}(1+\lambda g_{ij}) \tag{2.6}$$

$$\log_{1+\lambda}(1+\lambda g_{.j})=\sum_{i=1}^{n}\log_{1+\lambda}(1+\lambda g_{ij}) \tag{2.7}$$

Proposition 2.1 Let $(X\times Y, P(X\times Y), g_\lambda)$ be a product g_λ-measure space. Then the set function $g_\lambda^{(1)}: P(X)\to[0, 1]$ and $g_\lambda^{(2)}: P(Y)\to[0, 1]$ defined by

$$g_\lambda^{(1)}(A)=\frac{1}{\lambda}\cdot\left[\prod_{x_i\in A}(1+\lambda g_{i.})-1\right]\qquad \forall\ A\in P(X)$$

$$g_\lambda^{(2)}(B)=\frac{1}{\lambda}\cdot\left[\prod_{y_j\in B}(1+\lambda g_{.j})-1\right]\qquad \forall\ B\in P(Y)$$

are the g_λ-measures on $(X, P(X))$ and $(Y, P(Y))$ [5,6], respectively.

Proof. Obviously.

Proposition 2.2 Let $(X\times Y, P(X\times Y), g_\lambda)$ be a product g_λ-measure space. For an arbitrary fixed i $(1\leq i\leq n)$ with $g_{i.}\neq 0$, if let

$$g_{j|i}=\frac{1}{\lambda}\cdot\left[(1+\lambda)^{\log_{1+\lambda}(1+\lambda g_{ij})/[\log_{1+\lambda}(1+\lambda g_{i.})]}-1\right] \tag{2.8}$$

Then $\qquad \prod_{j=1}^{m}(1+\lambda g_{j|i}) = 1+\lambda$

Proof. By (2.8) we have

$$\log_{1+\lambda}(1+\lambda g_{ij}) = \log_{1+\lambda}(1+\lambda g_{j|i})\cdot\log_{1+\lambda}(1+\lambda g_{i.}) \tag{2.9}$$

So it follows from (2.7) that

$$\sum_{j=1}^{m}\log_{1+\lambda}(1+\lambda g_{j|i}) = \sum_{j=1}^{m}[\log_{1+\lambda}(1+\lambda g_{ij})\ /\ \log_{1+\lambda}(1+\lambda g_{i.})]$$
$$= \log_{1+\lambda}(1+\lambda g_{i.})/\log_{1+\lambda}(1+\lambda g_{i.}) = 1$$

That is $\qquad \prod_{j=1}^{m}(1+\lambda g_{j|i}) = 1+\lambda$.

Definition 2.2 Let $(X\times Y, P(X\times Y), g_\lambda)$ be a product g_λ-measure space. If

$$\log_{1+\lambda}(1+\lambda g_{ij}) = \log_{1+\lambda}(1+\lambda g_{i.})\cdot\log_{1+\lambda}(1+\lambda g_{.j}) \tag{2.10}$$

Then g_λ is said to be independent with respect to X and Y, and for short, g_λ is independent.

Proposition 2.3 If g_λ is independent and $g_{i.}\neq 0$ $(1\leq i\leq n)$, then

$$\log_{1+\lambda}(1+\lambda g_{j|i})=\log_{1+\lambda}(1+\lambda g_{.j})\qquad (1\leq j\leq m)$$

Proof. Follows from (2.9) and (2.10) evidently.

For $1 < i < n$, $1 < j < m$, let

$$p_{ij} = \log_{1+\lambda}(1+\lambda g_{ij}), \qquad p_{i.} = \log_{1+\lambda}(1+\lambda g_{i.})$$

$$p_{.j} = \log_{1+\lambda}(1+\lambda g_{.j}), \quad p_{j|i} = \log_{1+\lambda}(1+\lambda g_{j|i}) \qquad (2.11)$$

and $\quad P = (p_{ij})_{n\times m}$.

Obviously, $p_{i.} = \sum_{j=1}^{m} p_{ij}$, $p_{.j} = \sum_{i=1}^{n} p_{ij}$, $p_{ij} = p_{j|i} p_{i.}$ $(1 < i < n,\ 1 < j < m)$

$$\sum_{i=1}^{n} p_{i.} = \sum_{j=1}^{m} p_{.j} = \sum_{i=1}^{n} \sum_{j=1}^{m} p_{ij} = 1$$

and $$\sum_{j=1}^{m} p_{j|i} = \sum_{j=1}^{m} (p_{ij}/p_{i.}) = p_{i.}/p_{i.} = 1$$

Proposition 2.4 g_λ is independent if and only if the arbitrary two rows (columms) in the matrix P are proportional.

Proof. By definition 2.2, the proof of necessity is obvious. All we have to prove is the sufficiency. Here we consider the case of rows in P.

Suppose the arbitrary two rows in P are proportional. Especially, every row is proportional to the first row. Let

$$(p_{i1}\ p_{i2}\ \dots\ p_{im}) = k_i (p_{11}\ p_{12}\ \dots\ p_{1m}) \quad (1 < i < n) \qquad (2.12)$$

Then $$P = (k_1,\ k_2\ \dots\ k_n)^T (p_{11}\ p_{12}\ \dots\ p_{1m}) \qquad (2.13)$$

Therefore $p_{i.} = \sum_{j=1}^{m} p_{ij} = k_i \sum_{j=1}^{m} p_{1j} = k_i p_{1.}$ $(1 < i < n)$.

i.e. $$k_i = p_{i.}/p_{1.} \quad (1 < i < n) \qquad (2.14)$$

Consequently,

$$p_{.j} = \sum_{i=1}^{n} p_{ij} = \sum_{i=1}^{n} k_i p_{1j} = p_{1j} \sum_{i=1}^{n} k_i = p_{1j} \sum_{i=1}^{n} p_{i.}/p_{1.} \quad (1 < j < m)$$

i.e. $\quad p_{1j} = p_{1.} p_{.j}$ $(1 < j < m)$

Hence it follows from (2.13), (2.14) and (2.15) that

$$P = (1\ \ p_{2.}/p_{1.}\ \dots\ p_{n.}/p_{1.})^T (p_{1.}p_{.1}\ \ p_{1.}p_{.2}\ \dots\ p_{1.}p_{.m})$$
$$= (p_{1.}\ \ P_{2.}\ \dots\ p_{n.})^T (p_{.1}\ p_{.2}\ \dots\ p_{.m})$$

This means $\quad p_{ij} = p_{i.} p_{.j}$ $(1 < i < n,\ 1 < j < m)$.

i.e. $\quad \log_{1+\lambda}(1+\lambda g_{ij}) = \log_{1+\lambda}(1+\lambda g_{i.}) \log_{1+\lambda}(1+\lambda g_{.j})$

Thus g_λ is independent. The proof is commplete.

3. The Entropy of Product g_λ-measure Space

In his paper[5], Kruse introduced the concept of entropy of g_λ-measure space to measure the "fuzziness". In this section, we studied the entropy of product g_λ-measure space.

Definition 3.1 Let $(X\times Y,\ P(X\times Y),\ g_\lambda)$ be a product g_λ-measure space. The quantity

$$H_{nm}(\{g_{ij}\}) = -\sum_{i=1}^{n}\sum_{j=1}^{m}\log_{1+\lambda}(1+\lambda g_{ij})\log[\log_{1+\lambda}(1+\lambda g_{ij})] \quad (3.1)$$

(assume $0\cdot\log 0 = 0$)

is called the entropy of $(X\times Y,\ P(X\times Y),\ g_\lambda)$.

For a fixed i with $g_{i.}\neq 0$ $(1<i<n)$, the quantity

$$H_m(\{g_{j|i}\}) = -\sum_{j=1}^{m}\log_{1+\lambda}(1+\lambda g_{j|i})\log[\log_{1+\lambda}(1+\lambda g_{j|i}) \quad (3.2)$$

is called the entropy of $(Y,\ P(Y),\ g_\lambda^{(2)})$ given $x=x_i$.

If $g_{i.}\neq 0$ for $i=1,2,\ldots n$, then the average

$$H_X(Y) = \sum_{i=1}^{n}H_m(\{g_{j|i}\})\log_{1+\lambda}(1+\lambda g_{i.}) \quad (3.3)$$

is called the conditional entropy of $(X\times Y,\ P(X\times Y),\ g_\lambda)$ given X.

Obviously,
$$H_{nm}(\{g_{ij}\}) = -\sum_{i=1}^{n}\sum_{j=1}^{m}p_{ij}\log p_{ij}$$
$$H_m(\{g_{j|i}\}) = -\sum_{j=1}^{m}p_{j|i}\log p_{j|i}$$
$$H_X(Y) = \sum_{i=1}^{n}H_m(\{g_{j|i}\})p_{i.}$$

Where p_{ij}, $p_{i.}$, $p_{.j}$ and $p_{j|i}$ are given by (2.11).

Proposition 3.1 $H_{nm}(\{g_{ij}\})$ is decomposiable, i.e.

$$H_{nm}(\{g_{ij}\}) = H_n(\{g_{i.}\}) + H_X(Y) \quad (3.4)$$

Where $H_n(\{g_{i.}\})=-\sum_{i=1}^{n}p_{i.}\log p_{i.}$ is the entropy of $(X,\ P(X),\ g_\lambda^{(1)})$ [5].

Proof.
$$\begin{aligned}H_{nm}(\{g_{ij}\}) &= -\sum_{i=1}^{n}\sum_{j=1}^{m}p_{ij}\log p_{ij}\\
&= -\sum_{i=1}^{n}\sum_{j=1}^{m}p_{j|i}p_{i.}\log(p_{j|i}p_{i.})\\
&= -\sum_{i=1}^{n}\sum_{j=1}^{m}p_{j|i}p_{i.}\log p_{i.} - \sum_{i=1}^{n}\sum_{j=1}^{m}p_{j|i}p_{i.}\log p_{i.}\\
&= -\sum_{i=1}^{n}[p_{i.}\log p_{i.}(\sum_{j=1}^{m}p_{j|i})] - \sum_{i=1}^{n}[p_{i.}(\sum_{j=1}^{m}p_{j|i}\log p_{j|i})]\\
&= -\sum_{i=1}^{n}p_{i.}\log p_{i.} - \sum_{i=1}^{n}H_m(\{g_{j|i}\})p_{i.} = H_n(\{g_{i.}\}) + H_X(Y).\end{aligned}$$

Proposition 3.2 If g_λ is independent, then

$$H_X(Y) = H_m(\{g_{.j}\}) \quad (3.5)$$

Where $H_m(\{g_{.j}\}) = -\sum_{j=1}^{m}p_{.j}\log p_{.j}$ is the entropy of $(Y,\ P(Y),\ g_\lambda^{(2)})$ [5].

Proof. Since g_λ is independent, by proposition 2.3 we get $p_{j|i} = p_{.j}$ $(1<j<m)$. Therefore

$$\begin{aligned}H_X(Y) &= \sum_{i=1}^{n}H_m(\{g_{j|i}\})p_{i.} = \sum_{i=1}^{n}(\sum_{j=1}^{m}p_{j|i}\log p_{j|i})p_{i.}\\
&= \sum_{i=1}^{n}(\sum_{j=1}^{m}p_{.j}\log p_{.j})p_{i.} = (\sum_{i=1}^{n}p_{i.})\sum_{j=1}^{m}p_{.j}\log p_{.j})\\
&= \sum_{j=1}^{m}p_{.j}\log p_{.j} = H_m(\{g_{j|i}\}).\end{aligned}$$

Combining proposition 3.1 with proposition 3.2, we have

Corollary 3.1 If g_λ is independent, then

$$H_{nm}(\{g_{ij}\}) = H_n(\{g_{i.}\}) + H_m(\{g_{.j}\}) \qquad (3.6)$$

In general case, we have

Proposition 3.3 $\quad H_X(Y) \leq H_m(\{g_{.j}\}) \qquad (3.7)$

Proof. Let $\varphi(x)=-x\log x$ $(0<x\leq 1)$ and $\varphi(0)=0$. Then $\varphi(x)$ is a convex function on [0, 1]. It follows from the properties of convex function that for every $x_i \in [0, 1]$ and each $\alpha_i>0$ with $\sum_{i=1}^n \alpha_i=1$,

$$\sum_{i=1}^n \alpha_i \varphi(x_i) \leq \varphi(\sum_{i=1}^n \alpha_i x_i) \qquad (3.8)$$

Especially, subtituting $\alpha_i=p_{i.}$ and $x_i=p_{j|i}$ (i=1,2,...n) in (3.8) we have

$$-\sum_{i=1}^n p_{i.}p_{j|i}\log p_{j|i} \leq -(\sum_{i=1}^n p_{i.}p_{j|i})\log(\sum_{i=1}^n p_{i.}p_{j|i})$$

$$= -(\sum_{i=1}^n p_{ij})\log(\sum_{i=1}^n p_{ij}) = -p_{.j}\log p_{.j}$$

Hence $\quad H_X(Y) = \sum_{i=1}^n H_m(\{g_{j|i}\})\log_{1+\lambda}(1+\lambda g_{i.})$

$$= \sum_{i=1}^n(-\sum_{j=1}^m p_{j|i}\log p_{j|i})p_{i.} = -\sum_{j=1}^m\sum_{i=1}^n p_{i.}p_{j|i}\log p_{j|i}$$

$$\leq -\sum_{j=1}^m p_{.j}\log p_{.j} = H_m(\{g_{j.}\}).$$

Proposition 3.4. Let $1/n^*=\lambda^{-1}[(1+\lambda)^{1/n}-1]$, $1/m^*=\lambda^{-1}[(1+\lambda)^{1/m}-1]$.

Then $\quad H_{nm}(\{g_{ij}\}) \leq H_n(1/n^*, 1/n^* \ldots 1/n^*) + H_m(1/m^*, 1/m^* \ldots 1/m^*)$

Proof. $H_n(1/n^*, 1/n^* \ldots 1/n^*)$

$$= -\sum_{i=1}^n \log_{1+\lambda}(1+\lambda/n^*)\log[\log_{1+\lambda}(1+\lambda/n^*)]$$

$$= -\sum_{i=1}^n 1/n.\log(1/n) = \log n.$$

Similarly, $H_m(1/m^*, 1/m^* \ldots 1/m^*) = \log m$.

Applying (3.4), (3.7) and considering (3.8), we have

$$H_{nm}(\{g_{ij}\}) = H_n(\{g_{i.}\}) + H_X(Y) \leq H_n(\{g_{i.}\}) + H_m(\{g_{.j}\})$$

$$= \sum_{i=1}^n p_{i.}\log(1/p_{i.}) + \sum_{j=1}^m p_{.j}\log(1/p_{.j})$$

$$\leq \log[\ (p_{i.}/p_{i.})] + \log[\ (p_{.j}/p_{.j})]$$

$$= \log n + \log m = H_n(1/n^*, 1/n^* \ldots 1/n^*) + H_m(1/m^*, 1/m^* \ldots 1/m^*).$$

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