On the pseudo-null-additivity and the pseudo-autocontinuity of fuzzy measures

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Abstract: In this paper, the concept of pseudo-atom is introduced, and the equivalent property of the pseudo-null-additivity and the pseudo-autocontinuous is studied when the fuzzy measure is finite and the set X is countable.
Key Words: Fuzzy measure; psedo-null-addtivity; pseudo-autocontinuity.

1. Preliminaries

Let X be a nonempty set, $\mathfrak{F}$ is a σ-algebra of the subsets of X and $\mu: \mathfrak{F} \to [0, \infty)$ be a fuzzy measure, i.e. with monotonity and continuity and $\mu(\varphi)=0$. Throughout this paper, fuzzy measure space $(X, \mathfrak{F}, \mu)$ is fixed. For convenience, we at first recall some definitions and propositions on fuzzy measures.

Definition 1 ([2]) A fuzzy measure is called pseudo-null-additive, if $\mu(B \cup C) = \mu(C)$ whenever $A \in \mathfrak{F}$, $B \in A \cap \mathfrak{F}$, $C \in A \cap \mathfrak{F}$ with $\mu(A-B) = \mu(A)$.

Proposition 1 ([1]) The following statementes are pairwise equivalent:
(1). μ is pseudo-null-additive;
(2). $\mu(B \cap C) = \mu(C)$ whenever $A \in \mathfrak{F}$, $B, C \in A \cap \mathfrak{F}$ with $\mu(A) = \mu(B)$;
(3). $\mu(C \cup (A-B)) = \mu(C)$ whenever $A \in \mathfrak{F}$, $B, C \in A \cap \mathfrak{F}$ with $\mu(A) = \mu(B)$.

Definition 2 ([2]) A fuzzy measure μ is called pseudo- autocontinuous from above (resp. pseudo-autocontinuous from below), if for every $A \in \mathfrak{F}$ and $\{B_n\} \subset A \cap \mathfrak{F}$,

$$\mu(B_n) \to \mu(A) \qquad \mu((A-B_n) \cup C) \to \mu(C)$$
$$(\text{resp. } \mu(B_n \cap C) \to \mu(C))$$

for every $C \in A \cap \mathfrak{F}$; μ is said to be pseudo-autocontinuous, if it is both pseudo-autocontinuous from above and from below.

Proposition 2 ([1]). For a fuzzy measure, pseudo- autocontinuity from below (or from above) implies pseudo-null-additivity.

Proposition 3 ([1]). Let μ be a fuzzy measure, then μ is pseudo-autocontinuous from above if and only if μ is pseudo- autocontinuous from below.

Definition 3. Given a nonempty class of sets $\zeta \subset A \cap \mathfrak{F}$, for every $x \in \bigcup_{B \in A - \zeta} B$, denote $A_\zeta(x) = \cap \{B: x \in B \subset A - \zeta\}$. $A_\zeta(x)$ is called the pseudo-atom on x of ζ.

Proposition 4. If C is a pseudo-atom on x of ζ, then for every $B \in A - \zeta$, $x \in B$ implies $C \subset B$.

Proof. It is obvious by the Definition 3.

2. Main results

Lemma 1. If X is countable, then for every $A_n \in A \cap \mathfrak{F}$ $(n=1, 2, \ldots)$, $\limsup(A - A_n) = \bigcup_{t \in T} A_t$. Where $A_t \in A \cap \mathfrak{F}$ are pseudo-atom of $\zeta = \{A_n\} \subset A \cap \mathfrak{F}$ and T is a countable or finite index set.

Proof. Since X is countable, we may assume that

$$\limsup(A - A_n) = \{x_t: t \in T\},$$

T is a countable or finite index set. As for every $x_t \in \bigcup_{B \in A - \zeta} B$, there exists a pseudo-atom A_t on x_t of ζ, such that

$$\limsup(A - A_n) \subset \bigcup_{t \in T} A_t.$$

On the other hand, for every $t \in T$ and $x_t \in \limsup(A - A_n)$, there exists a subsequence $\{A_{t_n}\}$ of $\{A_n\}$ such that $x_t \in A - A_{t_n}$ $(n=1, 2, \ldots)$. By the Proposition 4 and $A_t = A(x_t)$ we have that

$$A_t \subset \limsup(A - A_n),$$

so that

$$\bigcup_{t \in T} A_t \subset \limsup(A - A_n).$$

This shows that

$$\limsup(A - A_n) = \bigcup_{t \in T} A_t.$$

Theorem 1. Let X be a countable set, and $\mu: \mathfrak{F} \to [0, \infty)$ be a fuzzy measure, then the pseudo-null-additivity is equivalent to pseudo-autocontinuity.

Proof. By the proposition 2 and 3, it suffices to show that the pseudo-null-additivity implies the pseudo-autocontinuity from above under the hypothesis. Suppose that the conclusion is false, then there exist $A \in \mathfrak{F}$, $\varepsilon_0 > 0$ and $B_n, C \in A \cap \mathfrak{F}$ $n=1, 2, \ldots$ with $\mu(B_n) \to \mu(A)$ such that

$$\mu(C \cup (A - B_n)) > \mu(C) + \varepsilon_0. \qquad (*)$$

Since $\mu(B_n)$ converges to $\mu(A)$. First, we have that

$$\mu(A - \limsup(A - B_n)) = \mu(A).$$

By the pseudo-null-additivity and (*), we have

$$\begin{aligned} \mu(C) &= \mu(\limsup(A - B_n) \cup C) \\ &\geqslant \limsup((A - B_n) \cup C) \\ &\geqslant \limsup(\mu(C) + \varepsilon_0) \\ &= \mu(C) + \varepsilon_0. \end{aligned}$$

It is a contradiction. Hence we complete the proof of the theorem.

References

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