ůřn

Differential and integral mean value theorems of fuzzy functions

Dang Faning, Guo Shuangbing and Liu Wangjin
Department of Mathematics, Southwest Jiaotong University, Chengdou Sichuan, 610031,P,R,China.

Abstract: In this paper we introduce relationships of precedence or quasiequality between any two fuzzy numbers, then give maximum [minimum] value theorem and intermediate value theorem of continuous fuzzy functions on closed interval.We discuss necessary and sufficient condition of monotonic increasing [monotonic decreasing] fuzzy functions, and discuss necessary condition and two decision theoremes of extreme value.Finally we give the Rolle's mean value theorem,lagrange's mean value theorem and a integral mean value theorem of fuzzy functions.
Keywords: Fuzzy functions, derivative, extreme value, integrate.

The definitions of differentiate and integrate of fuzzy functions have been given in [1,2,4,5,6] and some properties of differentiate and integrate of fuzzy functions have been studied in them. But the properties of continuous fuzzy functions, extreme value of fuzzy functions, differential and integral mean value theorems of fuzzy functions haven't been studied so far. In this paper we discuss these questions.

We use parameterized triples $\{(s(r), t(r), r)|\ 0 \leqslant r \leqslant 1\}$ to represent fuzzy numbers μ, and introduce relationships of precedence $\propto$ or quasiequality $\approx$ between any two fuzzy numbers. Dubois [7] indicated that the derivative of fuzzy functions of [6] needn't be fuzzy numbers. So in this paper, we adopt the definitions of differentiate and integrate of fuzzy functions of [3] and [1] respectuely.

1. Fuzzy numbers

Definition1.1 [6] A fuzzy number is a fuzzy set $\mu: R \rightarrow I = [0,1]$ with the properties

(1) μ is upper semicontinuous,

(2) $\mu(x) = 0$, outside of some interval $[c, d]$,

(3) there are real numbers a and b, $c \leqslant a \leqslant b \leqslant d$ such that μ is increasing on $[c, a]$, decreasing on $[b, d]$, and $\mu(x) = 1$ for each $x \in [a, b]$.

We let F denotes the family of fuzzy numbers.

We can indentify a fuzzy number μ with the parameterized triples

$$\{(s(r), t(r), r)|\ 0 \leqslant r \leqslant 1\} \quad (1)$$

where $s(r)$ and $t(r)$ denote the left hand endpoint and right hand endpoint of r-level subsets of μ respectuely.

Let $T(\mu)=\int_0^1 r[s(r)+t(r)]dr$, $T(\mu)$ is a real number.

The ralationship between the parameterized triples and fuzzy numbers can be illustrated by theorem1.2.

Theorem1.2 [6] Suppose that $s: I\to R$ and $t: I\to R$ satisfy the conditions

(1) s is a bounded increasing function,

(2) t is a bounded decreasing function,

(3) $s(1)\leq t(1)$,

(4) for $0<k\leq 1$, $\lim_{r\to k^-} s(r)=s(k)$ and $\lim_{r\to k^-} t(r)=t(k)$,

(5) $\lim_{r\to 0^+} s(r)=s(0)$ and $\lim_{r\to 0^+} t(r)=t(0)$.

Then $\mu: R\to I$ defined by

$$\mu(x)=\sup\{r \mid s(r)\leq x\leq t(r)\}$$

is a fuzzy number with parameterication given by (1). Moreover, if $\mu: R\to I$ is a fuzzy number with parametrization given by (1), then the functions s and t satisfy conditions (1) - (5).

Definition1.3 [8] A fuzzy number μ is positive fuzzy number if $\mu(x)=0$, for each $x\leq 0$.

A fuzzy number μ is negative fuzzy number if $\mu(x)=0$, for each $x\geq 0$.

Definition1.4 [8] Suppose $\mu=\{(s(r),t(r),r) \mid 0\leq r\leq 1\}$ and $\nu=\{(p(r), q(r),r) \mid 0\leq r\leq 1\}$ are any two fuzzy numbers. The operator $*$ is defined by

$$(\mu * \nu)(z)=\bigvee_{z=x*y}(\mu(x)\wedge\nu(y))$$

where $*$ can represent $+,-,\times$ or $\div$.

The proof of following formulas are straightforward.

$\mu+\nu=\{(s(r)+p(r),\ t(r)+q(r),r) \mid 0\leq r\leq 1\}$,

$\mu-\nu=\{(s(r)-q(r),\ t(r)-p(r),r) \mid 0\leq r\leq 1\}$,

$\mu\times\nu=\{(e(r),\ f(r),r) \mid 0\leq r\leq 1\}$,

where $e(r)=\min(s(r)p(r),s(r)q(r),t(r)p(r),t(r)q(r))$,

$f(r)=\max(s(r)p(r),s(r)q(r),t(r)p(r),t(r)q(r))$,

$\mu\div\nu$ is a fuzzy number when ν is a positive or negative fuzzy number, and

$\mu\div\nu=\{(e(r),f(r),r) \mid 0\leq r\leq 1\}$,

where $e(r)=\min(s(r)/p(r),s(r)/q(r),t(r)/p(r),t(r)/q(r))$,

$f(r)=\max(s(r)/p(r),s(r)/q(r),t(r)/p(r),t(r)/q(r))$.

Suppose k is a real number, we define the scalar product by

$$k\mu=\{(ks(r),kt(r),r) \mid 0\leq r\leq 1\}.$$

Definition1.5 [6] Suppose $\mu=\{(s(r),t(r),r) \mid 0\leq r\leq 1\}$, $\nu=\{(p(r),q(r),r) \mid 0\leq r\leq 1\}$ are any two fuzzy numbers.Then μ precedes ν ($\mu\propto\nu$) if

$$T(\mu)=\int_0^1 r[s(r)+t(r)]dr\leq T(\nu)=\int_0^1 r[p(r)+q(r)]dr$$

The character of difinition1.5 is any two elements in F are comparable under $\propto$, and give less importance to the lower levels of fuzzy numbers when compare the two elements in F.

Definition1.6 The two fuzzy numbers μ and ν are called quasiequality ($\mu \approx \nu$) if

$$T(\mu)=\int_0^1 r[s(r)+t(r)]\,dr = T(\nu)=\int_0^1 r[p(r)+q(r)]\,dr.$$

Lemma1.7 Suppose M is a real number, then at less exist one fuzzy number μ, so that $T(\mu)=M$.

Proof. Let $\mu(x)=1$, if $x=M$; $\mu(x)=0$, otherwise. then $T(\mu)=M$.

2. Properties of continuous fuzzy functions

Definition2.1 [3] A function $f: R\to F$, $X\to f(x)$ is said to be a fuzzy function. $f(x)$ can be represented parametrically by

$$\{(s(r,x),\ t(r,x),r)\mid 0\leqslant r\leqslant 1\}$$

Let $$T(f(x))=\int_0^1 r[s(r,x)+t(r,x)]\,dr.$$

$T(f(x))$ is a real function.

Definition2.2 [3] Suppose that $f: R\to F$ is a fuzzy function and that for each x, $f(x)$ is represented parametrically by $\{(s(r,x),\ t(r,x),r)\mid 0\leqslant r\leqslant 1\}$. $f(x)$ is called continuous fuzzy function on R if for any $r\in[0,1]$, both $s(r,x)$ and $t(r,x)$ are continuous real functions on R.

Lemma2.3 If $f(x)$ is a continuous fuzzy function on some interval, then $T(f(x))$ is a continuous function.

Proof. It is straightforward from the definition of $T(f(x))$.

Theorem2.4 Suppose $f(x)$ and $g(x)$ are two continuous fuzzy functions at $x'\in R$, then $f(x)\pm g(x)$, $f(x)\times g(x)$ are continuous fuzzy functions at same point x'. $f(x)\div g(x)$ is a continuous fuzzy function if $g(x)$ is positive or negative fuzzy function and $g(x)\not\approx \mathbf{0}$. Where $\mathbf{0}=\{(0,0,r)\mid 0\leqslant r\leqslant 1\}$.

Proof. Immediately from definition1.4 and definition2.2.

Theorem2.5 [Maximin value theorem] Suppose $f:[a,b]\to F$ is a continuous fuzzy function, then $f(x)$ can get maximum [minimum] value on $[a,b]$.

Proof. From lemma2.3, we are sure $T(f(x))$ is continuous on $[a,b]$, so $T(f(x))$ can get maximum [minimum] value on $[a,b]$. Frome definition1.5 the maximum [minimum] value point of $T(f(x))$ exact the maximum [minimum] value point of $f(x)$.

Theorem2.6 [Bounded theorem] If fuzzy function $f[a,b]\to F$ is continuous on $[a,b]$, then $f(x)$ is bounded on $[a,b]$. That is, exist fuzzy numbers μ' and μ'', such that $\mu'\propto f(x)\propto \mu''$ for any $x\in[a,b]$.

Proof. $T(f(x))$ is continuous on $[a,b]$ from lemma2.3. So exist real numbers M' and M'', such that $M'\leqslant T(f(x))\leqslant M''$ for any $x\in[a,b]$. From lemma1.7, exist fuzzy numbers μ' and μ'', so that $T(\mu')=M'$ and $T(\mu'')=M''$. Then from definition1.5, we can sure $\mu'\propto f(x)\propto \mu''$ satisfy demand of the theorem.

Theorem2.7 [Intermediate value theorem] Suppose $f:[a,b]\to F$ is a continuous fuzzy function, μ' and μ'' are maximum value and minimum value of $f(x)$ on $[a,b]$ respectuely, then for any ν, $\mu' \propto \nu \propto \mu''$, exist $\xi \in [a,b]$, such that $f(\xi) \approx \nu$.

Proof. From definition1.5,we can be sure $T(\mu')$ and $T(\mu'')$ are maximum value and minimum value of $T(f(x))$ on $[a,b]$ respectuely, and $T(\mu') \leqslant T(\nu) \leqslant T(\mu'')$, so exist $\xi \in [a,b]$, such that $T(f(\xi))=T(\nu)$. From definition1.6, $f(\xi) \approx \nu$.

3. Differentiate and extreme value of fuzzy functions

Definition3.1 [3] Suppose $f(x)=\{(s(r,x),t(r,x),r) \mid 0\leqslant r\leqslant 1\}$ is a fuzzy function which is defined on R. We say $f(x)$ is derivable on R if both $s(r,x)$ and $t(r,x)$ are derivable on R for any $r\in[a,b]$, and $f'(x)$ is defined by

$$f'(x)=\bigcup_{r\in[0,1]} r\,[\min(s_x(r,x),t_x(r,x)),\ \max(s_x(r,x),t_x(r,x))]$$

$f'(x)$ is a fuzzy function also. Where $s_x(r,x)$ and $t_x(r,x)$are the partial derivatives of $s(r,x)$ and $t(r,x)$ respect to x.

Definition3.2 [3] Suppose $f(x)=\{(s(r,x),t(r,x),r) \mid 0\leqslant r\leqslant 1\}$ is a fuzzy function which is defined on $[a,b]$. $f(x)$ is said to preserve order derivable [reverse order derivable] on $[a,b]$ if both $s(r,x)$ and $t(r,x)$ are derivable on $[a,b]$ for any $r\in[0,1]$, $s_x(r,x)\leqslant t_x(r,x)$ $[t_x(r,x)\leqslant s_x(r,x)]$,and $[s_x(r',x),t_x(r',x)]\geqslant[s_x(r'',x),t_x(r'',x)]$ when $r'\leqslant r''$.

Definition3.3 Suppose $f'(x)$ is a derived function of fuzzy function $f(x)$. If $f'(x)$ is derivable on $[a,b]$,then the derived function of $f'(x)$is called the second order derived function of $f(x)$ on $[a,b]$, represent by $f''(x)$.

$f''(x)$ is called second preserving order derived [reversing order derived] function of $f(x)$ if $f'(x)$ is preserving order derived [reversing order derived] function of $f(x)$ and $f''(x)$ is preserving order deriverd [reversing order derived] function of $f'(x)$.

Notation3.4 In order to discuss conveniencely, all derivable functions what we use in the following are preserving order derivable functions, and

$$f'(x)=\{(s_x(r,x),t_x(r,x),r) \mid 0\leqslant r\leqslant 1\}.$$

Definition3.5 [6] A fuzzy function $f:R\to F$ is monotonic increasing [monotonic decreasing] if

$$f(x')\propto f(x''),\ \text{when}\ x'\leqslant x''\quad [f(x'')\propto f(x'),\ \text{when}\ x'\leqslant x''].$$

Definition3.6 Suppose $f:[a,b]\to F$ is a continuous fuzzy function. $f(x)$ is said to have maximum [minimum] value $f(y)$ at point $y\in[a,b]$ if exist a neighbourhood $U(y)$, such that for any $x\in U(y)$, $f(x)\propto f(y)$ $[f(y)\propto f(x)]$. y is called maximum [minimum] value point.Both maximum [minimum] value are called extreme value.

Lemma3.7 (1) [6] A fuzzy function $f: R \to F$ is monotonic increasing [monotonic decreasing] on $[a, b]$ if and only if $T(f): R \to R$ is monotonic increasing [monotonic decreasing] on $[a, b]$.

(2) If $f(x)$ is derivable on some interval, then $T(f(x))$ is derivable on same interval.

(3) $0 \propto f'(x)$ [$f'(x) \propto 0$] if and only if $0 < T'(f(x))$ [$T'(f(x)) < 0$].

Theorem3.8 Suppose $f: R \to F$ is deriveable at every points, then $f(x)$ is monotonic increasing [monotonic decreasing] if and only if $0 \propto f'(x)$ [$f'(x) \propto 0$].

Proof. From lemma3.7, for any $x \in R$, $T(f(x))$ is derivable. $f(x)$ is monotonic increasing [monotonic decreasing] if and only if $T(f(x))$ is monotonic increasing [monotonic decreasing], if and only if $0 < T'(f(x))$ [$T'(f(x)) < 0$], from lemma3.7, $0 < T'(f(x))$ [$T'(f(x)) < 0$] if and only if $0 \propto f'(x)$ [$f'(x) \propto 0$].

Theorem3.9 Suppose fuzzy function $f(x)$ have definition on the neighbourhood $U(y)$, $f(x)$ is derivable and get maximum value or minimum value at point y, then $f'(y) \approx 0$.

Proof. From the conditions of this theorem, we know that $T(f(x))$ have definition on the neighbourhood $U(y)$, and $T(f(x))$ is derivable and get maximum value or minimum value at point y. So $0 = T'(f(y)) = \int_0^1 r(s_x(r, y) + t_x(r, y))dr$, from definition1.6, we have $f'(y) \approx 0$.

Theorem3.10 [Necessary condition of extreme value] If y is extreme point of fuzzy function $f(x)$, then y is null point of $f'(x)$ or inderivable point of $f(x)$.

Proof. For y is extreme point of $f(x)$, we can be sure y is extreme point of $T(f(x))$, So y is null point of $T'(f(x))$ or inderivable point of $T(f(x))$. From definition1.6, the null point of $T'(f(x))$ is null point of $f'(x)$, and from $T(f(x)) = \int_0^1 r(s(r, x) + t(r, x))dr$ is inderivable at point y, we can get at less one of $s(r, x)$ and $t(r, x)$ is in derivable at point y, so $f(x)$ is inderivable at point y.

Theorem3.11 [Decision theorem of extreme value I] Suppose fuzzy function $f(x)$ is derivable on $(y-\delta, y)$ and $(y, y+\delta)$ (where $\delta > 0$), then

(1) y is minimum value point if $f'(x) \propto 0$ for every $x \in (y-\delta, y)$ and $0 \propto f'(x)$ for every $x \in (y, y+\delta)$;

(2) y is maximum value point if $0 \propto f'(x)$ for every $x \in (y-\delta, y)$ and $f'(x) \propto 0$ for every $x \in (y, y+\delta)$.

(3) y is not extreme point if $f'(x)$ preserve symbol on $(y-\delta, y)$ and $(y, y+\delta)$.

Proof. The proof is straightforward from theorem3.8.

Theorem3.12 [Decision theorem of extreme value II] Suppose $f(x)$ is a fuzzy function, $f'(y)\approx 0$,

(1) $f(y)$ is maximum value if $f''(y)\propto 0$;

(2) $f(y)$ is minimum value if $0\propto f''(y)$.

Proof. (1) Suppose $f(x)=\{(s(r,x),t(r,x),r)\mid 0\leqslant r\leqslant 1\}$, then $f'(x)=\{(s_x(r,x),t_x(r,x),r)\mid 0\leqslant r\leqslant 1\}$, $f''(x)=\{(s_{xx}(r,x),t_{xx}(r,x),r)\mid 0\leqslant r\leqslant 1\}$, from $f'(y)\approx 0$ and $f''(y)\propto 0$, we have $T'(f(y))=0$ and $T''(f(y))\leqslant 0$, so y is maximum value point of $T(f(x))$, then $f(y)$ is maximum value of $f(x)$.

(2) The proof is similar to (1).

4. Differential mean value theorems of fuzzy functions

Theorem4.1 [Rolle's mean value theorem] Suppose fuzzy function $f(x)$ satisfy the conditions

(1) $f(x)$ is continuous on $[a,b]$,

(2) $f(x)$ is derivable on (a,b),

(3) $f(a)\approx f(b)$,

then at less exist one point $\xi\in(a,b)$, such that $f'(\xi)\approx 0$.

Proof. From lemma3.7 and conditions of this theorem we have $T(f(x))$ satisfies the conditions of crisp Rolle's mean value theorem, so at less exist one point $\xi\in(a,b)$, such that $T'(f(\xi))=0$. From definition1.6 we know $f'(\xi)\approx 0$.

Theorem4.2 [Lagrange's mean value theorem] Suppose fuzzy function $f(x)$ satisfy the conditions

(1) $f(x)$ is continuous on $[a,b]$,

(2) $f(x)$ is derivable on (a,b),

then at less exist one point $\xi\in(a,b)$, such that

$$f'(\xi)\approx[f(b)-f(a)]/(b-a)$$

Proof. From lemma3.7 and conditions of this theorem we have $T(f(x))$ satisfies the conditions of crisp Lagrange's mean value theorem, so at less exist one point $\xi\in(a,b)$, such

$$T'(f(\xi))=[T(f(b))-T(f(a))]/(b-a)$$
$$=[\int_0^1 r(s(r,b)+t(r,b))dr-\int r(s(r,a)+t(r,a))dr]/(b-a)$$
$$=\int_0^1 r\{[s(r,b)-t(r,a)]/(b-a)+[t(r,b)-s(r,a)]/(b-a)\}dr$$

from definition1.6 we know

$$f'(\xi)\approx[f(b)-f(a)]/(b-a).$$

Corollary4.3 Suppose $f(x)$ is a fuzzy function, and $f'(x)\approx 0$ for any $x\in(a,b)$, then $f(x')\approx f(x'')$ for any $x',x''\in(a,b)$.

Proof. For any two point $x',x''\in(a,b)$, suppose $x'<x''$, from Lagrange's mean value theorem we have $f(x'')-f(x')\approx f'(\xi)(x''-x')$, but $f'(\xi)\approx 0$, therefore $f(x'')\approx f(x')$.

Corollary4.4 Suppose $f(x)$ and $g(x)$ are two fuzzy function, and $f'(x)\approx g'(x)$ for any $x\in(a,b)$, then $f(x)\approx g(x)+\mu$ for any $x\in(a,b)$ (where μ is a fuzzy number).

Proof. The proof is immediate from corollary4.3.

5. Integral mean value theorem of fuzzy functions

Definition5.1 [1] Suppose $f(x)=\{(s(r,x),t(r,x),r)\mid 0\leq r\leq 1\}$ is a fuzzy function, we call $f(x)$ is integrable on $[a,b]$ if both $s(r,x)$ and $t(r,x)$ are integrable on $[a,b]$ for any $r\in[0,1]$. The integration of $f(x)$ is defined by

$$\int_a^b f(x)\,dx=\bigcup_{r\in[0,1]} r\left[\int_a^b s(r,x)\,dx,\int_a^b t(r,x)\,dx\right].$$

$\int f(x)\,dx$ is a fuzzy function also.

Theorem5.2 Suppose $f(x)$ is a continuous fuzzy function on $[a,b]$, then exist $\xi\in[a,b]$, such that

$$\int_a^b f(x)\,dx\approx f(\xi)(b-a).$$

Proof. Since $f(x)$ is continuous on $[a,b]$, from maximum value theorem2.5 we know, exist two fuzzy numbers m and M, such that $m\propto f(x)\propto M$ for any $x\in[a,b]$, then $m(b-a)\propto\int f(x)\,dx\propto M(b-a)$. Now from intermediate value theorem2.7 we have, exist $\xi\in[a,b]$, such that $\int_a^b f(x)\,dx\approx f(\xi)(b-a)$.

6. Concluding remarks

Especially, if the fuzzy function $f(x)$ is a crisp function, the results of this paper are also satisfactory, and these results accord with the theory of crisp function.

References

[1] Luo Chengzhong, Wang Demou, Extension of the integral of interval-valued function and the integral of fuzzy valued function, Fuzzy Mathematics, 3 (1983) 45-52.

[2] Wang Demou, Luo Chengzhong, Extension of fuzzy differential calculus, Fuzzy Mathematics, 1 (1985) 75-80.

[3] Cen Yongting, Newton-Leibniz formulas of interval-valued function and fuzzy-valued function. Fuzzy Mathematics, 3.4 (1987) 13-18.

[4] D.Dubois and H.Prade, Towards fuzzy differential calculus, Part 3: Differentiation, Fuzzy Sets and Systems, 8 (1982) 225-235.

[5] M.L.Puri and D.A.Ralescu, Differential of fuzzy functions, J. Math. Anal. Appl. 91 (1983) 552-558.

[6] R.Goetschel Jr. and W.Voxman, Elementary fuzzy calculus, Fuzzy Sets and Systems, 18(1986) 31-43.

[7] D.Dubois and H.Prade, On several definitions of the differential of a fuzzy mapping, Fuzzy Sets and Systems, 24 (1987) 117-120.

[8] Zhang Wenxou, Wang Guogun, Liu Wangjin, Fang Jinxuan, Introduction of theory of fuzzy mathematics, Published In Xian Jiaotong University, 1991,6.