ON THE CONTINUITY OF FUZZY NUMBER VALUED FUNCTION

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Abstract: In this paper, we show that the fuzzy number valued function on n-dimensional fuzzy number space $F_{cc}(R^n)$, induced by a uniformly continuous real number valued function on n-dimensional Euclidean space R^n through Zadeh's extension priciple, is continuous with respect to Haudorff metrics on $F_{cc}(R^n)$ and $F_{cc}(R^1)$.

Keywords: fuzzy number, Hausdorff metric, continuous function

The theory and applications to mathematics and real-problems of fuzzy numbers have been studied by many researchers. Some of these studies can be found in the papers listed in the References of this paper.

In this paper, we consider the problems concerned with the continuity of fuzzy number valued function on n-dimensional fuzzy number space. We show that the fuzzy number valued function F on n-dimensional fuzzy number space $F_{cc}(R^n)$, induced by a uniformly continuous real number valued function f on R^n through Zadeh's extension principle, is continuous with respect to Hausdorff metrics on $F_{cc}(R^n)$ and $F_{cc}(R^1)$.

Throughout this paper, R^n is the n-dimensional Euclidean space, the continuity of the function from R^n to R^1 is in the sense of Euclidean metrics on R^n and R^1: $d_E[R^n]$ and $d_E[R^1]$.

Definition 1. An n-dimensional fuzzy number (1-dimensional fuzzy number is simply called fuzzy number) is a fuzzy set on R^n with the membership function

$\mu_A(x)$ which satisfies

(1) $A_1 = \{x \in R^n;\ \mu_A(x) = 1\} \neq \emptyset$;

(2) $A_\alpha = \{x \in R^n;\ \mu_A(x) \geq \alpha\}$ ($\alpha \in (0, 1]$) is a convex, compact subset of R^n (i.e., A_α is a convex, bounded closed subset of R^n) with respect to Euclidean metric $d_E[R^n]$.

We denote $F_{cc}(R^n) = \{A;$ A is an n-dimensional fuzzy number$\}$.

Definition 2. $\forall$ A, $B \in F_{cc}(R^n)$, the Hausdorff metric $d_H[F_{cc}(R^n)]$ of A, B is defined via

$$d_H[F_{cc}(R^n)](A, B) = \bigvee_{\alpha \in (0,1]} d_H[R^n](A_\alpha, B_\alpha)$$
$$= \bigvee_{\alpha \in (0,1]} \max\left\{\max_{x \in A_\alpha} d_E[R^n](x, B_\alpha),\ \max_{y \in B_\alpha} d_E[R^n](A_\alpha, y)\right\},$$

where $d_H[R^n]$ is the Hausdorff metric on the class of all compact subsets on R^n.

If n = 1, we have

$$d_H[F_{cc}(R^1)](A, B) = \bigvee_{\alpha \in (0,1]} [|A_\alpha^- - B_\alpha^-| \vee |A_\alpha^+ - B_\alpha^+|],$$

where $A^\pm$, $B^\pm$ are determined via

$$A_\alpha = [A_\alpha^-, A_\alpha^+]$$

and

$$B_\alpha = [B_\alpha^-, B_\alpha^+].$$

In this paper, we obtain

Theorem 1. If $f: R^n \to R^1$ is a uniformly continuous function, then the function $F: F_{cc}(R^n) \to F_{cc}(R^1)$, induced by f through Zadeh's extension principle, is continuous with respect to Hausdorff metrics on $F_{cc}(R^n)$ and $F_{cc}(R^1)$.

PROOF. First, we show that $\forall A \in F_{cc}(R^n)$, $F(A) \in F_{cc}(R^1)$.

Indeed, from Nguyen [7], we know that $\forall \alpha \in (0, 1]$,

$$[F(A)]_\alpha = f(A_\alpha)$$

and then

$$[F(A)]_1 = f(A_1) \neq \emptyset.$$

Since A_α is compact, then so is $[F(A)]_\alpha = f(A_\alpha)$.

$\forall f_1, f_2 \in [F(A)]_\alpha = f(A_\alpha)$, $f_1 < f_2$, there exist $x_1, x_2 \in A_\alpha$ such that

$$f(x_i) = f_i,$$

$i = 1, 2$.

As f is continuous on the connected subset $K = \{x \in R^n;\ x = tx_1 + (1-t)x_2,\ 0 \leq t \leq 1\}$, and, the convexity of K implies that $K \subset A_\alpha$, therefore, $\forall \hat{f} \in [f_1, f_2]$, there exists $\hat{x} \in K \subset A_\alpha$ such that

$$f(\hat{x}) = \hat{f}$$

which shows $[F(A)]_\alpha = f(A_\alpha)$ is convex.

Second, we prove that F is continuous with respect to Hausdorff metrics on $F_{cc}(R^n)$ and $F_{cc}(R^1)$.

$\forall \varepsilon > 0$, since f is uniformly continuous, then there exists $\delta = \delta(\varepsilon) > 0$ such that $\forall x, y \in R^n$, $d_E[R^n](x, y) \leq \delta$ implies

$$d_E[R^1](f(x), f(y)) = |f(x) - f(y)| \leq \varepsilon.$$

As $\forall A \in F_{cc}(R^n)$, $[F(A)]_\alpha = f(A_\alpha)$ is convex and compact, then it is a bounded closed interval on R^1, and it can be written by

$$[F(A)]_\alpha = f(A_\alpha) = [\min_{x \in A_\alpha} f(x),\ \max_{x \in A_\alpha} f(x)].$$

For $\varepsilon > 0$ given above and $A, B \in F_{cc}(R^n)$, if

$d_H[F_{cc}(R^n)](A, B) \leq \delta$, we can verify that

$$d_H[F_{cc}(R^1)](F(A), F(B)) \leq \varepsilon.$$

Indeed, from the fact that

$$d_H[F_{cc}(R^1)]\ (F(A), F(B)) = \bigvee_{\alpha \in (0,1]} d_H[R^1]([F(A)]_\alpha, [F(B)]_\alpha)$$

$$= \bigvee_{\alpha \in (0,1]} d_H[R^1](f(A_\alpha), f(B_\alpha)),$$

we know that it is sufficient to show that $\forall \alpha \in (0, 1]$,

$$d_H[R^1](f(A_\alpha), f(B_\alpha)) \leq \varepsilon,$$

i. e.,

$$d_H[R^1]([\min_{x_1 \in A_\alpha} f(x_1), \max_{y_1 \in A_\alpha} f(y_1)], [\min_{x_2 \in B_\alpha} f(x_2), \max_{y_2 \in B_\alpha} f(y_2)]) \leq \varepsilon,$$

or equivalently,

$$\left| \min_{x_1 \in A_\alpha} f(x_1) - \min_{x_2 \notin B_\alpha} f(x_2) \right| \leq \varepsilon \tag{i}$$

and

$$\left| \max_{y_1 \in A_\alpha} f(y_1) - \max_{y_2 \notin B_\alpha} f(y_2) \right| \leq \varepsilon \tag{ii}$$

We only give the proof of (i). The rest is similar.

Denote $C_\alpha = A_\alpha \cup B_\alpha$. Then C_α is a bounded closed subset of R^n.

(I) If the minimum of f on C_α reaches at $\bar{x} \in A_\alpha$ and $\bar{y} \in B_\alpha$, we have

$$\min_{x_1 \in A_\alpha} f(x_1) = \min_{x_2 \in B_\alpha} f(x_2) = \min_{x \in C_\alpha} f(x)$$

and hence

$$\left| \min_{x_1 \in A_\alpha} f(x_1) - \min_{x_2 \in B_\alpha} f(x_2) \right| \leq \varepsilon$$

(II) If the minimum of f on C_α reaches at $\bar{x} \in A_\alpha - B_\alpha$ and $\forall x \in B_\alpha$, $f(x) > f(\bar{x}) = \min_{x \in C_\alpha} f(x)$, we can select $\bar{y} \in B_\alpha$ such that $f(\bar{y}) = \min_{x_2 \in B_\alpha} f(x_2)$, and then

$$f(\bar{x}) = \min_{x_1 \in A_\alpha} f(x_1) < \min_{x_2 \in B_\alpha} f(x_2) = f(\bar{y}) \tag{iii}$$

Since

$$d_H[R^n](A_\alpha, B_\alpha) \leq d_H[F_{cc}(R^n)](A, B) \leq \delta$$

we have, for $\bar{x} \in A_\alpha - B_\alpha$,

$$d_E[R^n](\bar{x}, B_\alpha) \leq \delta,$$

which yields that there exists $\bar{z} \in B_\alpha$ such that

$$d_E[R^n](\bar{x}, \bar{z}) = d_E[R^n](\bar{x}, B_\alpha) \leq \delta,$$

and therefore,

$$0 \leq f(\bar{z}) - f(\bar{x}) \leq \varepsilon \tag{iv}$$

(iii) and (iv) imply that

$$0 < \min_{x_2 \in B_\alpha} f(x_2) - \min_{x_1 \in A_\alpha} f(x_1) = f(\bar{y}) - f(\bar{x}) \leq f(\bar{z}) - f(\bar{x}) \leq \varepsilon,$$

and hence,

$$\left| \min_{x_1 \in A_\alpha} f(x_1) - \min_{x_2 \in B_\alpha} f(x_2) \right| \leq \varepsilon .$$

(III) If the minimum of f on C_α reaches at $\bar{x} \in B_\alpha - A_\alpha$ and $\forall x \in A_\alpha$, $f(x) > f(\bar{x})$, we can similarly show that

$$\left| \min_{x_1 \in A_\alpha} f(x_1) - \min_{x_2 \in B_\alpha} f(x_2) \right| \leq \varepsilon .$$

The conclusions obtained in (I), (II) and (III) ensure that (i) holds, and the proof is complete.

Lastly, we point out that, in Theorem 1, if f is replaced by a function f from R^n to R^m ($m > 1$), it may appear that $F(A) \notin F_{cc}(R^m)$ for $A \in F_{cc}(R^n)$, i. e., f may not induce a function from $F_{cc}(R^n)$ to $F_{cc}(R^m)$, as the following example indicated.

Example 1. Let $f: R^2 \rightarrow R^2$ be

$$f(x) = f(y, z) = (y, \sin y)$$

and let $A \in F_{cc}(R^2)$ with

$$\mu_A(x) = 1, \quad \text{if } x = (y, z) \in [-\pi, \pi] \times [-1, 1],$$
$$= 0, \quad \text{else.}$$

Then

$$F(A)(x) = 1, \quad \text{if } x = (y, z) \in \{(y, \sin y);\ -\pi \leq y \leq \pi\},$$
$$= 0, \quad \text{else,}$$

and hence, $\forall \alpha \in (0, 1]$,

$[F(A)]_\alpha = \{(y, \sin y);\ -\pi \leq y \leq \pi\}$ is not convex, and therefore,

$F(A) \notin F_{cc}(R^2)$.

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