56

LOGICAL ANALYSIS OF MAX–MIN RULE OF INFERENCE

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Abstract

This paper presents a logical analysis of the well known Max–Min rule of inference. Though non–sound in general, it is possible to restrict the domain of this rule to be defined only for formulas having disjoint antecedents so that it becomes sound. Furthermore, it is a fuzzy interpretation of a disjunction of conjunctions that describe a function. The Max-Min rule of inference is thus less general than the inference rules based on modus ponens.

1 Introduction

In the theory of approximate reasoning, two trends can be distinguished. First, most common trend, understands approximate reasoning as a special theory of fuzzy relations. The inference, motivated by logic, is modelled as a composition of fuzzy relations and new special operations are searched to improve its behaviour to meet some intuitive requirements. Logical aspects are taken into account only partly.

The second trend is formally logical and tries to comprehend and describe approximate reasoning as a theory of human thinking which uses natural language expressions and follows logical laws.

The most widely used rule of inference is the Max–Min one. This was originally proposed by E. H. Mamdani who based it on ideas of L. A. Zadeh presented in his famous paper [12]. The basis of this rule is interpretation of the linguistically expressed rule

$$\text{IF } X \text{ is } \mathscr{A} \text{ THEN } Y \text{ is } \mathscr{B} \tag{1}$$

as a Cartesian product

$$A \times B \tag{2}$$

where A, B are fuzzy sets assigned to the linguistic expressions $\mathscr{A}, \mathscr{B}$, respectively as their meaning. In his paper [12], L. A. Zadeh understood (2) as "$\mathscr{A}$ relates to $\mathscr{B}$" which seems to be reasonable. Nevertheless, in fuzzy control, the relation (2) remained an interpretation of implication statement (1) despite the fact that Cartesian product is symmetric operation and thus, it cannot serve as a model of implication.

When more rules (1) are given then their aggregation leads to a fuzzy relation

$$R = \bigcup_{i=1}^{m} R_i = \bigcup_{i=1}^{m} (A_i \times B_i). \tag{3}$$

When a statement "X is $\mathscr{A}'$" is given where $\mathscr{A}'$ is interpreted by a fuzzy set A' then we obtain the well-known Max–Min rule of inference

$$B'y = A'x \wedge \bigvee_{i=1}^{m} (A_i x \times B_i y) \tag{4}$$

which is the basis of fuzzy control.

In this paper, we will analyze this rule from a more formal point of view.

2 Fuzzy logic and Max–Min rule

Since logic deals with truth values and a membership degree Ax can be interpreted as a truth value of the statement "x belongs to A", we may see that approximate reasoning deals with sets of truth values. This was elaborated more formally in [7] and [8]. Thus, deriving a conclusion in approximate reasoning means making many partial inferences in many–valued (fuzzy) logic.

One of crucial notions in logic is that of a model. Informally, a model is a mathematical abstraction of a reality. Of course, there may exist many models. A many–valued rule of inference is a couple

$$r : \frac{A_1, \dots, A_n}{r^{syn}(A_1, \dots, A_n)} \left(\frac{a_1, \dots, a_n}{r^{sem}(a_1 \dots, a_n)} \right) \tag{5}$$

where r^{syn} is a syntactical operation assigning a formula $r^{syn}(A_1, \dots, A_n)$ to formulas $A_1, \dots, A_n$ and r^{sem} is a semantic operation applied to the truth valuations $a_i \in L$ of the respective formulas $A_i, i = 1, \dots, n$.

A rule is ***sound***, i.e., it cannot lead to incorrect results which would not reflect the reality if

$$\mathscr{D}(r^{syn}(A_1, \dots, A_n)) \geq r^{sem}(\mathscr{D}(A_1), \dots, \mathscr{D}(A_1)) \tag{6}$$

holds for every model $\mathscr{D}$. Roughly speaking, if (6) is not fulfilled then we can derive facts (formulas) that might not represent the reality. Note that syntax in fuzzy logic is evaluated, i.e., correctness of each step in the derivation is evaluated by some truth value. This is a direct generalization of classical logic where, of course, only steps evaluated by 1 (truth) are considered (zero is uninteresting and other values are impossible).

From the point of view of fuzzy logic, Max–Min rule of inference can be rewritten into a form of a many–valued inference rule as follows

$$r_{DC} : \frac{A_k(x), \bigvee_{j=1}^{m}(A_j(x) \wedge B_j(y))}{B_k(y)} \left(\frac{a, b}{a \wedge b}\right), \qquad 1 \leq k \leq m. \tag{7}$$

In approximate reasoning, we have to consider sets of formulas $\{A_x[t];\ t \in M_V\}$ where M_V is a set of all terms without variables. Testing the soundness of (7) leads to the requirement

$$\mathscr{D}(B_{k,y}[s]) \geq \mathscr{D}(A_{k,x}[t]) \wedge \bigvee_{i=1}^{m}(\mathscr{D}(A_{i,x}[t]) \wedge \mathscr{D}(B_{i,y}[s])) \tag{8}$$

for every model $\mathscr{D}$ and terms $t, s \in M_V$. However, it may easily be demonstrated that there is a combination of a model $\mathscr{D}$ and t, s such that (8) does not hold. Inspecting it in more details shows that (8) is violated in the cases when

$$A_{k,x}[t] \wedge A_{i,x}[t] > 0 \tag{9}$$

and

$$B_{k,y}[s] < B_{i,y}[s], \qquad i \neq k.$$

Thus, we may overcome the non-soundness of (7) when accepting that the respective fuzzy sets A_i occurring in antecedents of IF–THEN rules (1) are mutually disjoint, i.e.,

$$A_i \cap A_j = \emptyset, \qquad i \neq k. \tag{10}$$

This conclusion is in accordance with [3] where a weaker conditions is imposed, namely

$$A_i \otimes A_j = \emptyset, \qquad i \neq k \tag{11}$$

where $\otimes$ denotes bold intersection of fuzzy sets (cf. [5]).

Proposition 1 *Let a syntactic part of the rule r_{DC} be defined for all couples of formulas of the form $\langle A_k, \bigvee_{i=1}^{m}(A_i \wedge B_i)\rangle$, $1 \leq k \leq m$ such that $A_i \wedge B_j \equiv \mathbf{o}$ for all $i \neq j$. Then the rule r_{DC} (7) is sound.*

PROOF: Let $\mathscr{D}$ be a model. Then

$$\mathscr{D}(B_k) \geq \mathscr{D}(A_k) \wedge \bigvee_{i=1}^{m} \mathscr{D}(A_i \wedge B_i) = \mathscr{D}(A_k) \wedge \mathscr{D}(B_k)$$

since $\bigvee_{\substack{i=1\\i\neq k}}^{m} (\mathscr{D}(A_k) \wedge \mathscr{D}(A_i) \wedge \mathscr{D}(B_i)) = 0$ due to the assumption on the domain of r^{syn}. □

Let us now demonstrate the danger of non–soundness of Max–Min rule.

Example 1 Let a temperature t of a furnace be controlled by an amount of fuel dependently on the position a of a regulator cock: 'turn to the right' means increase, and 'turn to the left' decrease of the amount of fuel. Assume the control process is given by the following two rules[†)]:

IF t is *big* THEN a is *negatively big*
IF t is *medium* THEN a is *zero*

The terms are interpreted by the fuzzy sets depicted on Figure 1.

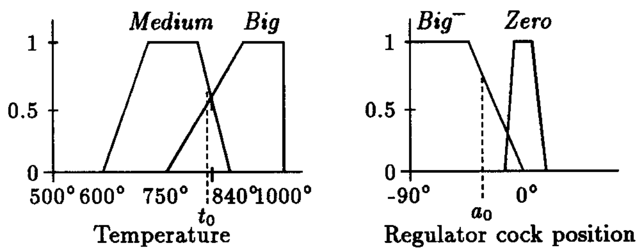

Figure 1: Shapes of fuzzy sets.

The condition (6) for the rule (7) can be rewritten as follows:

$$Pos(a) \geq Temp(t) \wedge ((Big(t) \wedge Big^-(a)) \vee (Medium(t) \wedge Zero(a))) \quad (12)$$

where $Pos(a), Temp(t)$ stand for the respective formulas representing the position and temperature. For all the values $t \in (750°, 840°)$ of temperature $Temp(a) \equiv Medium(a)$ should hold (i.e., the given temperature t should be evaluated as "medium") and due to the above defined rules we would expect that $Pos(a) \equiv Zero(a)$. However,

$$Zero(a) < Medium(t) \wedge ((Big(t) \wedge Big^-(a)) \vee (Medium(t) \wedge Zero(a))),$$

for all a such that $Zero(a) < Big^-(a)$. Consequently, the condition (12) is not fulfilled.

†)More rules can (and should) be easily added but we do not need them for our purpose.

Let, for example, the input temperature $t_0 = 825°$. Due to the above defined rules, this temperature should be evaluated as medium and thus, we expect not to turn the regulator cock very much. However, due to the rules and using the center of gravity defuzzification method we obtain $a_0 \approx -42°$ which is quite much to the left. Consequently, the control action is incorrect.

In practice, this incorrectness is tuned out by changing the shape of fuzzy sets no matter how they reflect the intuitive meaning of the given words. However, this is not in accordance with the proclaimed intentions of approximate reasoning. Human comprehension of words corresponds to the reality and he does not change it when making his reasoning.

At the end of this section, let us note the form of the rules (1). They are stated as linguistically expressed logical implications. However, (2) is, in fact, a conjunction and thus, Max-Min rule is an interpretation of the disjunction

$$\begin{array}{ll} \textsf{IF } X \textsf{ is } \mathscr{A}_1 \textsf{ AND } Y \textsf{ is } \mathscr{B}_1 & \textsf{OR} \\ \dots\dots\dots\dots\dots\dots\dots\dots & \textsf{OR} \\ \textsf{IF } X \textsf{ is } \mathscr{A}_1 \textsf{ AND } Y \textsf{ is } \mathscr{B}_1 & \end{array} \tag{13}$$

The original IF–THEN form seems to be motivated by the following proposition. Let $A^i(x) := x = a_i$ and $B^i(y) := y = b_i$ be formulas of classical logic where $a_1, \dots, a_m, b_1, \dots, b_m$ are terms without variables and F be a function symbol such that

$$\vdash F(a_i) = b_i \tag{14}$$

for every $i = 1, \dots, m$.

Proposition 2

$$\vdash \bigwedge_{i=1}^{m} (A_x^i[t] \Rightarrow B_y^i[F(t)]) \equiv \bigvee_{i=1}^{m} (A_x^i[t] \wedge B_y^i[F(t)]) \tag{15}$$

for every term $t \in \{a_1, \dots, a_m\}$.

PROOF: Let $t = a_j$, i.e., $A_x^j[t]$ is true. Then $A_x^i[t]$ is not true for all $i \neq j$ and, at the same time, $F(a_j) = b_j$ due to (14), i.e., $B_y^j[F(a_j)]$ is true. Hence, the left side of (15) is true. But then also the right side is true since $A_x^j[a_j]$ as well as $B_y^j[F(a_j)]$ are true. □

It follows from this proposition that in classical logic, we may describe a function either as a disjunction of conjunctions or as a conjunction of implications. However, this cannot be stated in fuzzy logic. Indeed, let us take $A^i(x)$ and $B^i(y)$ as fuzzy equalities. Then (14) is provable only in a certain degree and thus, (15) should be considered for any couple of terms t, s, i.e., we would have to obtain

$$\bigwedge_{i=1}^{m} \mathscr{D}(A_x^i[t] \Rightarrow B_y^i[s]) = \bigvee_{i=1}^{m} \mathscr{D}(A_x^i[t] \wedge B_y^i[s])$$

for any model $\mathscr{D}$ and $t, s \in M_V$. However, since $\mathscr{D}(A_x^i[t])$ has, in general, nothing common with $\mathscr{D}(B_y^i[s])$ (t, s are taken quite arbitrarily), this equality cannot be obtained. We conclude that if we want to use Max–Min rule of inference, we must consider the set of linguistic conditions to be given only in the form (13). The IF–THEN form is rather a licence based on analogy from classical logic.

3 Conclusion

The Max–Min rule of inference is the most often used rule in approximate reasoning and especially in fuzzy control inferences. In this paper, we have demonstrated that it leads to several problems. A consistent use of this rule must fulfil the following conditions:

(a) The linguistic rules representing the description of the control process should be understood as conjunctions joined by disjunction (13) being a linguistic description of a *function*. They cannot be understood as real IF–THEN rules.

(b) The fuzzy sets considered for the values of the independent variables must be pairwise disjoint. Hence, they cannot be fuzzy interpretations of the meaning of general linguistic terms such as "small", "big", etc., but rather that of fuzzy numbers.

If these conditions are fulfilled then Max–Min rule does not lead to incorrect results and it is acceptable from the logical point of view. This result is also in accordance with [2] and [3]. If we want to understand IF–THEN rules indeed as implications which thus would describe a more general free relation between dependent and independent variables, and if we want to consider the $\mathscr{A}, \mathscr{B}$ as linguistic terms interpreted properly we must use other rules of inference based on modus ponens in fuzzy logic — see [7, 8, 9].

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