CLOSURE OPERATORS: AN EXTENSION PRINCIPLE

by

Giangiacomo Gerla

Dipartimento di Matematica ed Applicazioni *R.Caccioppoli*

Monte S. Angelo, via Cinzia 80126 Napoli, Italy.

1. Introduction.

The notions of closure operator and closure system are very useful tools in several sections of classical mathematics. As an example, we may quote the closure systems and the related closure operators given by, the class of closed subsets of a given topological space, the class of substructures of a given algebraic structure, the class of filters of a given Boolean algebra, the class of convex subsets of a given Euclidean space. This leaded several authors to investigate about the closure systems and the closure operators in the framework of fuzzy set theory. As an example, see Biacino and Gerla [1984] , Murali [1991], Biacino [1993].

Let S be any set, denote by U the unitary real interval and recall that a *fuzzy subset* of S any map s:S→U (see Zadeh [1965]). We say that s is *crisp* provided that $s(x) \in \{0,1\}$ for every $x \in S$. We indicates by $\mathcal{F}(S)$ the class of the fuzzy subsets of S and by $\mathcal{P}(S)$ the class of the subsets of S, respectively. We identify $\mathcal{P}(S)$ with the class of crisp fuzzy subsets of S, namely we identify any $X \in \mathcal{P}(S)$ with the related characteristic function χ_X.

In this paper we propose an "extension principle" enabling to extend any classical operator $J:\mathcal{P}(S)\to\mathcal{P}(S)$ to an operator $J^*:\mathcal{F}(S)\to\mathcal{F}(S)$ in such a way that J is a closure operator if and only if J^* is a closure operator. Also, we examine a related "extension principle" to extend a class $\mathcal{C}$ of subsets of S in a class $\mathcal{C}^*$ of fuzzy subsets of S in such a way that $\mathcal{C}$ is a closure system if and only if $\mathcal{C}^*$ is a closure system. This enables us to restate in an uniform way several basic notions in fuzzy set theory, such as the ones of natural fuzzy topology, fuzzy subalgebra, necessity measures, fuzzy convex subsets and so on. Moreover, the application of such extension principles to the of classical deductive systems suggests the formulation of a promising class of fuzzy logics. We omit the proofs that can be founded in Gerla [1993].

2. Preliminaries.

In the sequel we adopt the convention that, if $(S,\leq,0,1)$ is any ordered set with minimum 0 and maximum 1, then least upper bound of the empty class is 0 and the greatest lower bound is 1, that is $Sup(\emptyset)=0$ and $Inf(\emptyset)=1$. If λ_1 and λ_2 are elements of U, we set $\lambda_1 \vee \lambda_2 = \mathrm{Max}\{\lambda_1, \lambda_2\}$ and $\lambda_1 \wedge \lambda_2 = \mathrm{Min}\{\lambda_1, \lambda_2\}$. Given $s, s' \in \mathcal{F}(S)$, we say that s is *contained* in s' and we write $s \subseteq s'$ provided that $s(x) \leq s'(x)$ for every $x \in S$. Also, the *union* $s \cup s'$ is defined by setting $(s \cup s')(x) = s(x) \vee s'(x)$ and the *intersection* $s \cap s'$ by setting $(s \cap s')(x) = s(x) \wedge s'(x)$ for every $x \in S$. More generally, given a family $(s_i)_{i \in I}$ of fuzzy subsets of S, we set

$$\left(\bigcup_{i \in I} s_i\right)(x) = \mathrm{Sup}\{s_i(x) / i \in I\} \quad \text{and} \quad \left(\bigcap_{i \in I} s_i\right)(x) = \mathrm{Inf}\{s_i(x) / i \in I\}.$$

Finally, the *complement* $-s$ of s is defined by $-s(x) = 1 - s(x)$. Given a fuzzy subset s of S, for every $\lambda \in U$ the subset $C(s,\lambda) = \{x \in S \mid s(x) \geq \lambda\}$ is called the *closed λ-cut* of s and

$O(s,\lambda) = \{x \in S \mid s(x) > \lambda\}$ is called the *open λ-cut* of s. The *support* Supp(s) of s is defined by $\mathrm{Supp}(s) = \{x \in S \mid s(x) \neq 0\}$. We say that s is *finite* if Supp(s) is finite. The following, are the main properties of the cuts

a) $C(s \cup s', \mu) = C(s,\mu) \cup C(s',\mu)$; $O(s \cup s', \mu) = O(s,\mu) \cup O(s',\mu)$

b) $C(s \cap s', \mu) = C(s,\mu) \cap C(s',\mu)$; $O(s \cap s', \mu) = O(s,\mu) \cap O(s',\mu)$

c) $C(s,\mu) = \bigcap_{\lambda < \mu} O(s,\lambda)$; $O(s,\mu) = \bigcup_{\lambda > \mu} C(s,\lambda)$

d) $C(s, \mathrm{Sup}_{i \in I} \lambda_i) = \bigcap_{i \in I} C(s,\lambda_i)$; $O(s, \mathrm{Inf}_{i \in I} \lambda_i) = \bigcup_{i \in I} O(s,\lambda_i)$

e) $C(\bigcap s_i, \lambda) = \bigcap C(s_i,\lambda)$; $O(\bigcup s_i, \lambda) = \bigcup O(s_i,\lambda)$.

where $s, s' \in \mathcal{F}(S)$, $\lambda, \mu \in U$, $(\lambda_i)_{i \in I}$ is a family of elements of U and $(s_i)_{i \in I}$ is a family of fuzzy subsets. A fuzzy subset is characterized by its cuts, indeed

$$s(x) = \mathrm{Sup}\{\lambda \in U \mid x \in C(s,\lambda)\}. \tag{2.1}$$

$$s(x) = \mathrm{Sup}\{\lambda \in U \mid x \in O(s,\lambda)\}. \tag{2.2}$$

More generally, any family $(C_\lambda)_{\lambda \in U}$ of subsets of S defines a fuzzy subset as follows

$$s(x) = \mathrm{Sup}\{\lambda \in U \mid x \in C_\lambda\}. \tag{2.3}$$

Lemma 2.1 Let $(C_\lambda)_{\lambda \in U}$ be any decreasing family of subsets of S, and define s by (2.3), then, for every $\mu \in U$

$$O(s,\mu) = \bigcup_{\lambda > \mu} C_\lambda \subseteq C_\mu \subseteq \bigcap_{\lambda < \mu} C_\lambda = C(s,\mu). \tag{2.4}$$

We call *continuous* any chain $(C_\lambda)_{\lambda \in U}$ of subsets of S such that $C_0 = S$ and $C_\mu = \bigcap_{\lambda < \mu} C_\lambda$. The following proposition shows that we may identify the fuzzy subsets of S with the continuous chains of subsets of S (see Negoita and Ralescu [1975]).

Proposition 2.2 Given a fuzzy subset s the family $(C(s,\lambda))_{\lambda \in U}$ of its closed cuts is a continuous chain. Conversely, given any continuous chain $(C_\lambda)_{\lambda \in U}$ of subsets of S define s by (2.3). Then s is a fuzzy subset such that $C(s,\mu) = C_\mu$ for every $\mu \in U$.

3. Closure operators.

Recall that, given a set S, a (classical) *closure operator* in $\mathcal{P}(S)$ is a map $J:\mathcal{P}(S)\to\mathcal{P}(S)$ such that, for every X and Y subsets of S,

$$\text{(i)}\quad X\subseteq Y \Rightarrow J(X)\subseteq J(Y)\qquad \text{(ii)}\quad X\subseteq J(X)\qquad \text{(iii)}\quad J(J(X))=J(X).$$

A collection $\mathcal{C}$ of subsets of S is *a closure system* if the intersection of any family of elements of $\mathcal{C}$ is an element of $\mathcal{C}$. In particular, since S is the intersection of the empty family, $S\in\mathcal{C}$. It is well known that if J is a closure operator then the set $\mathcal{C}_J=\{X \mid J(X)=X\}$ is a closure system and that if $\mathcal{C}$ is a closure system then, by setting $J_{\mathcal{C}}(X)=\bigcap\{Y\in\mathcal{C} \mid Y\supseteq X\}$ we obtain a closure operator $J_{\mathcal{C}}$.

To extend such concepts to fuzzy set theory, we call *fuzzy operator*, in brief *operator*, any map J from $\mathcal{F}(S)$ to $\mathcal{F}(S)$ and we say that J is a *fuzzy closure operator*, in brief a *closure operator*, provided that

$$\text{(i)}\quad s\subseteq s' \Rightarrow J(s)\subseteq J(s')\quad ;\quad \text{(ii)}\quad s\subseteq J(s)\quad \text{(iii)}\quad J(J(s))=J(s).$$

Likewise, a class $\mathcal{C}$ of fuzzy subsets of S is called a *fuzzy closure system*, in brief a *closure system*, if the intersection of any family of elements of $\mathcal{C}$ is an element of $\mathcal{C}$.

Proposition 3.1 A fuzzy closure system $\mathcal{C}$ is a complete lattice in which
- the meets coincide with the intersections in $\mathcal{F}(S)$;
- the join of a family $(s_i)_{i\in I}$ of elements of $\mathcal{C}$ is $\bigcap\{s\in\mathcal{C} \mid s\supseteq \bigcup s_i\}$.

The following proof, whose proof is immediate, shows that the notion of fuzzy closure system is strictly related with the one of fuzzy closure operator.

Proposition 3.2 Let $\mathcal{C}$ be a class of fuzzy subsets, then the operator $J_{\mathcal{C}}$ defined by

$$J_{\mathcal{C}}(s)=\bigcap\{s'\in\mathcal{C} \mid s'\supseteq s\} \tag{3.1}$$

is a fuzzy closure operator. Let J be any fuzzy closure operator and set

$$\mathcal{C}_J=\{f\in\mathcal{F}(S) \mid J(f)=f\} \tag{3.2}$$

then $\mathcal{C}_J$ is a closure system. Moreover, if J is a closure operator and $\mathcal{C}$ a closure system, then

$$J_{\mathcal{C}_J}=J \quad\text{and}\quad \mathcal{C}_{J_{\mathcal{C}}}=\mathcal{C}. \tag{3.3}$$

4. Extending classical closure operators.

Given a classical operator $J:\mathcal{P}(S)\to\mathcal{P}(S)$ we extend it in a fuzzy operator $J^*:\mathcal{F}(S)\to\mathcal{F}(S)$ by setting, for every $s\in\mathcal{F}(S)$

$$J^*(s)(x)=\mathrm{Sup}\{\lambda\in U \mid x\in J(C(s,\lambda))\} \tag{4.1}$$

We call *canonical extension* of J the operator J^*. From Lemma 2.1 it follows that

$$O(J^*(s),\mu) = \bigcup_{\lambda>\mu} J(C(s,\lambda)) \subseteq J(C(s,\mu)) \subseteq \bigcap_{\lambda<\mu} J(C(s,\lambda)) = C(J^*(s),\mu). \tag{4.2}$$

Proposition 4.1 Let J be a classical operator, then J^* is an extension of J. Moreover, J^* is a closure operator if and only if J is a closure operator. In this case the closed cuts of J^* are fixed points for J.

We may extend any class $\mathcal{C}$ of classical subsets of S in a class $\mathcal{C}^*$ of fuzzy subsets of S by setting

$$\mathcal{C}^*=\{s\in\mathcal{F}(S) \mid C(s,\lambda)\in\mathcal{C} \text{ for every } \lambda\in U,\ \lambda\neq 0\}. \tag{4.3}$$

Proposition 4.2 $\mathcal{C}$ coincides with the class of crisp elements of $\mathcal{C}^*$. Moreover, $\mathcal{C}^*$ is a fuzzy closure system if and only if $\mathcal{C}$ is a closure system.

We call $\mathcal{C}^*$ *the canonical extension* of $\mathcal{C}$ and we may also identify $\mathcal{C}^*$ with the class of the continuous chains of elements of $\mathcal{C}$.

Proposition 4.3 Let $\mathcal{C}$ be a classical closure system and $J_{\mathcal{C}}$ the classical closure operator generated by $\mathcal{C}$. Then

$$J_{\mathcal{C}^*}=(J_{\mathcal{C}})^* \tag{4.4}$$

5. Extending algebraic closure operators.

Recall that a classical closure operator $J:\mathcal{P}(S)\to\mathcal{P}(S)$ J is called *algebraic* if, for every subset X of S, $J(X)=\bigcup\{X_f \mid X_f \text{ is a finite part of } X\}$. A closure system $\mathcal{C}$ of subsets of S is called *algebraic* if the union of every chain of elements of $\mathcal{C}$ belongs to $\mathcal{C}$. It is immediate to prove that, for every closure system $\mathcal{C}$,

$$J_{\mathcal{C}} \text{ is algebraic } \Leftrightarrow \mathcal{C} \text{ is algebraic}$$

and that, for every closure operator J

$$J \text{ is algebraic } \Leftrightarrow \mathcal{C}_J \text{ is algebraic.}$$

In this section we will examine the canonical extension J^* of an algebraic operator J. In the sequel we will write $X\vdash_J x$ to denote that $x\in J(X)$ and, if $X=\{x_1,\dots,x_n\}$, we will write $x_1,\dots,x_n\vdash_J x$ instead of $\{x_1,\dots,x_n\}\vdash_J x$.

Proposition 5.1 Let $J:\mathcal{P}(S)\to\mathcal{P}(S)$ be algebraic then

$$J^*(v)(x)=\begin{cases}1 & \text{if } x\in J(\emptyset)\\ \mathrm{Sup}\{v(x_1)\wedge\dots\wedge v(x_n) \mid x_1,\dots,x_n\vdash_J x\} & \text{otherwise.}\end{cases} \tag{5.1}$$

6. Examples of canonical extensions.

In this section we will expose some examples of canonical extensions. To simplify our notations, in the sequel we assume that $\mathcal{C}$ is a classical closure system and that $J=J_{\mathcal{C}}$ the related closure operator. Notice that the proofs since are all rather immediate consequence of Proposition 5.1.

The natural fuzzy topologies. Famous examples of non algebraic classical closure systems are furnished by the class $\mathcal{C}$ of the closed subsets the topological spaces. Namely, if (S,τ) is a topological space, $\tau \subseteq \mathcal{P}(S)$, we denote by $\mathcal{C}$ the class of closed subsets and, for every subset X of S, by J(X) the set of points adherent to X. It is immediate to see that the canonical extension $\mathcal{C}^*$ of $\mathcal{C}$ coincides with the class of the upper semicontinuous fuzzy subsets. Now, the class $\tau^*=\{s \mid -s \in \mathcal{C}^*\}$ is a fuzzy topology that was examined in Conrad [1980] under the name of *natural fuzzy topology*. So, we can interpret $\mathcal{C}^*$ as the class of closed subsets of a fuzzy topology. As a consequence, since $J_{\mathcal{C}^*}=J^*$, while J(X) is the topological closure in τ of a subset X, $J^*(v)$ is the topological closure in τ^* of a fuzzy subset v.

Proposition 6.1 For every fuzzy subset v, the topological closure of v is given by
$$J^*(v)(x)=\mathrm{Sup}\{\mathrm{Inf}_{n\in N}v(x_n) \mid (x_n)_{n\in N} \text{ is a sequence s.t. } x=\lim x_n\}. \quad (6.1)$$

The natural fuzzy topologies enable us to to show that in (4.2) we cannot set the equality in the place of the inclusion, in general. Indeed, assume that $S=[0,1]$, consider the usual topology in S and let $v:S\to S$ be the fuzzy subset defined by setting $v(x)=x$ if $x\neq 1$ and $v(x)=0$ if $x=1$. Since $J(C(v,\lambda))=J([\lambda,1))=[\lambda,1]$ for every $\lambda\neq 1$, we have that
$$J^*(v)(x)=\mathrm{Sup}\{\lambda\in U \mid x\in J(C(v,\lambda))\}=\mathrm{Sup}\{\lambda\in U \mid \lambda\leq\mu\}=x,$$
and therefore that $J^*(v)$ is the identity map. Then, while $J(C(v,1))=J(\emptyset)=\emptyset$, we have $C(J^*(v),1)=\{1\}$ and while $J(O(v,0))=J(S)=S$ it is $O(J^*(v),0)=(0,1]$. Thus
$$C(J^*(v),\lambda)\neq J(C(v,\lambda)) \quad\text{and}\quad O(J^*(v),\lambda)\neq J(O(v,\lambda)).$$

Convex fuzzy subsets. Assume that $\mathcal{C}$ is the class of convex subsets of a Euclidean space $\mathbb{E}$ and therefore that, for every subset X of $\mathbb{E}$, J(X) is the convex envelope of X. Then, $\mathcal{C}^*$ is the class of *convex* fuzzy subsets as defined in Zadeh [1965] and $J^*(v)$ is the convex envelope of v.

Proposition 6.2 For every fuzzy subset v of an Euclidean space, the convex envelope of v is given by

$$J^*(v)(x)=\mathrm{Sup}\{v(x_1)\wedge\ldots\wedge v(x_n) \mid z=\lambda_1x_1+\ldots+\lambda_nx_n,\ \lambda_1,\ldots,\lambda_n\in U,\ \lambda_1+\ldots+\lambda_n=1\}\quad(6.2)$$

If $\mathbb{E}$ is the real line, the convex fuzzy subsets are known under the name of *convex fuzzy numbers*. Simple calculations enable us to prove that the convex fuzzy number $J^*(v)$ generated by v is given by

$$J^*(v)(z)=\mathrm{Sup}\{v(x_1)\wedge v(x_2) \mid x_1\le z\le x_2\}=\left(\mathrm{Sup}\{v(x_1) \mid x_1\le z\}\right)\wedge\left(\mathrm{Sup}\{v(x_2) \mid z\le x_2\}\right).$$

The generalized necessities. Assume that $\mathbb{C}$ is the class of filters of a Boolean algebra **B** and therefore that, for every subset X of **B**, J(X) is the filter generated by X. It is well known that $\mathbb{C}$ is an algebraic closure system and therefore that J is an algebraic closure operator. We call *fuzzy filters* the elements of $\mathbb{C}^*$ and therefore, for every fuzzy subset v of formulas, $J^*(v)$ is the *fuzzy filter generated by* v. Obviously, every filter is a fuzzy filter and, in particular, the whole algebra **B** (that is the map constantly equal to 1) is a fuzzy filter. Now, recall that a *generalized necessity* is any map $n:\mathbf{B}\to U$ such that

$$n(\mathbf{1})=1 \quad ; \quad n(x\wedge y)=n(x)\wedge n(y).$$

for every $x,y\in\mathbf{B}$ (see Biacino and Gerla [1992]). The name "generalized necessity" is justified by the fact that the generalized necessities n for which $n(\mathbf{0})=0$ are known in literature under the name of *necessities* (see Dubois and Prade [1988]).

Proposition 6.3 The fuzzy filters coincide with the generalized necessities.

Proposition 6.4 For every fuzzy subset v the fuzzy filter $J^*(v)$ generated by v is given by

$$J^*(v)(x)=\begin{cases}\mathrm{Sup}\{v(x_1)\wedge\ldots\wedge v(x_m) \mid x_1\wedge\ldots\wedge x_m\le x\} & \text{if } z\neq\mathbf{1}\\ 1 & \text{if } z=\mathbf{1}\ .\end{cases}\quad(6.3)$$

Let $(n_i)_{i\in I}$ be a nonempty family of necessities, then, since $\bigcap n_i$ is a generalized necessity and $(\bigcap n_i)(\mathbf{0})=\mathrm{Inf}\{n_i(\mathbf{0}) \mid i\in I\}=0$, $\bigcap n_i$ is a necessity. As a consequence, if $\mathcal{N}$ is obtained by adding to the set of the necessities the *inconsistent necessity* that is the map constantly equal to 1, then $\mathcal{N}$ is a closure system. Given a fuzzy subset v, we denote by $\overline{v}$ the necessity generated by v. We say that v is *inconsistent* if $\overline{v}$ is the inconsistent necessity and this is equivalent to say that a necessity exists containing v. If v is consistent then (6.3) gives the necessity generated by v. Indeed, if n is a necessity containing v, then since

$$y_1\wedge\ldots\wedge y_m=\mathbf{0}\Rightarrow v(y_1)\wedge\ldots\wedge v(y_m)\le n(y_1)\wedge\ldots\wedge n(y_m)=n(y_1\wedge\ldots\wedge y_m)=n(\mathbf{0})=0,$$

we have $J^*(v)(\mathbf{0})=0$ and therefore that $J^*(v)=\overline{v}$. ⊣

7. The case of the fuzzy subalgebras.

In the sequel $\mathcal{A}=(\mathbb{A},\mathbb{H},\mathbb{C})$ denotes an algebraic structure, where $\mathbb{A}$ is the domain, $\mathbb{H}$ is the set of operations on $\mathbb{A}$ and $\mathbb{C}\subseteq\mathbb{A}$ is the set of constants. Assume that $\mathcal{C}$ is the class of subalgebras of $\mathcal{A}$, where, if there is no constant the empty subset is considered as a subalgebra, then $\mathcal{C}$ is an algebraic closure system and, for every subset X of $\mathbb{A}$, J(X) is the subalgebra of $\mathcal{A}$ generated by X. In accordance with the literature, we write $<X>$ instead of J(X). Then, the elements of $\mathcal{C}^*$ are the fuzzy subsets whose closed cuts are subalgebras of $\mathcal{A}$ and are well known in literature under the name of *fuzzy subalgebras* (Rosenfeld [1971] and Di Nola and Gerla [1987]). Moreover, $J^*(v)$ is the fuzzy subalgebra generated by v and we denote it by $<v>$.

Proposition 7.1 Denote by Pol($\mathcal{A}$) the set of polynomial function of $\mathcal{A}$, then, for every fuzzy subset v, the fuzzy subalgebra generated by v is given by

$$<v>(x)=\begin{cases}\mathrm{Sup}\{v(x_1)\wedge\ldots\wedge v(x_n)\mid p(x_1,\ldots,x_n)=z\ ,\ p\in \mathrm{Pol}(\mathcal{A})\} & \text{if } x\notin <\mathbb{C}> \\ 1 & \text{if } x\in <\mathbb{C}>.\end{cases} \qquad (7.1)$$

Formula (7.1) becomes very simple if we consider classes of algebraic structures in which the polynomial function can be reduced to a canonical form. As an example, if $\mathcal{A}$ is a semigroup then the fuzzy subsemigroup $<v>$ generated by v is obtained by

$$<v>(z)=\begin{cases}\mathrm{Sup}\{v(x_1)\wedge\ldots\wedge v(x_n)\mid x_1\ldots x_n=z\} & \text{if } z\neq 1\\ 1 & \text{if } z=1\end{cases}$$

If $\mathcal{A}$ is a group then the fuzzy subsemigroup $<v>$ generated by v is given by

$$<v>(z)=\begin{cases}\mathrm{Sup}\{v(x_1)\wedge\ldots\wedge v(x_n)\mid x_1^{i_1}\cdot\ldots\cdot x_n^{i_n}=z\ ,\ i_1,\ldots,i_n\in\{1,-1\}\} & \text{if } z\neq 1\\ 1 & \text{if } z=1.\end{cases}$$

(see Biacino and Gerla [1984]). We conclude by noticing that further examples of fuzzy closure operators shoud be obtained by the free, pure, very pure, left unitary, right unitary, unitary fuzzy subsemigroups of a free semigroup $\mathcal{A}$ (see Gerla [1985])

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