On Convex Fuzzy Sets

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Abstract

This paper gives some properties of convex fuzzy sets, strictly convex fuzzy sets and strongly convex fuzzy sets.

1. Introduction

Convex fuzzy sets were first defined by Zadeh in [1]. Some properties were subsequently studies by Brown in [2], Weiss in [3], Katsaras and liu in [4], Lowen in [5], Liu in [6] and Yang in [7]. In this paper some properities of convex fuzzy sets, strictly convex fuzzy sets and strongly convex fuzzy sets are studied.

2. Notations

Throughout this paper E will denote the n-dimensional Euclidena space R^n. Fuzzy sets and values will be denoted by lower case Greek letters and we shall make no difference between notations for a fuzzy set with a constant value and that value itself. $I=(0,1)$

The fuzzy set λ on E is said to be convex fuzzy set if

$$\lambda(ax+(1-a)x) \geqslant \lambda(x) \wedge \lambda(y)$$

for every $x\in E$, $y\in E$ and $a\in I$.

The fuzzy set λ is said to be strongly convex fuzzy set if

$$\lambda(ax+(1-a)y) > \lambda(x) \wedge \lambda(y)$$

for every $x\in E$, $y\in E$, $x\neq y$ and $a\in I$.

The fuzzy set λ is said to be strictly convex fuzzy if

$$\lambda(ax+(1-a)y) > \lambda(x) \wedge \lambda(y)$$

for every $x\in E$, $y\in E$, $\lambda(x)\neq\lambda(y)$ and $a\in I$.

the fuzzy set λ on E is said to be a fuzzy closed set iff for all $a\in I$, $\lambda^{-1}[a,1]$ is closed.

3. Main Ressults

Theorem 1. let λ be a strictly fuzzy set on E, if there exists $a\in I$, for every $x, y\in E$ such that

$$\lambda(ax+(1-a)y) \geqslant \lambda(x) \wedge \lambda(y). \quad \text{(A)}$$

Then λ is convex fuzzy set on E.

Proof . By contradiction , suppose that there exists $x, y \in E$ and $b \in I$ such that

$$\lambda(bx+(1-b)y) < \lambda(x) \wedge \lambda(y).$$

Without loss of generality , assume that $\lambda(x) \leqslant \lambda(y)$ and let $z=bx+(1-b)y$.

$$\lambda(z) < \lambda(x) \wedge \lambda(y). \quad (1)$$

If $\lambda(x) < \lambda(y)$, since λ is strictly convex fuzzy set , we have

$$\lambda(z) > \lambda(x) \wedge \lambda(y).$$

contradicting (1).

If $\lambda(x) = \lambda(y)$, then (1) implies that

$$\lambda(z) < \lambda(x) = \lambda(y) \quad (2)$$

(I) If $0<b<a$, let $z_1=(b/a)x+(1-b/a)y$.

Thus ,

$$z=bx+(1-b)y=a[(b/a)x+(1-b/a)y]+(1-a)y=az_1+(1-a)y.$$

According to (A) , we have

$$\lambda(z) \geqslant \lambda(z_1) \wedge \lambda(y).$$

Since (2) and inequality above , then

$$\lambda(z) > \lambda(z_1) \quad (3)$$

Let $c=[(1-a)/a][b/(1-b)]$, because of $0<b<a<1$, it is easy to show $0<c<1$.

Thus ,

$$z_1=(b/a)x+(1-b/a)y=(b/a)x+(1-b/a)[z/(1-b)-bx/(1-b)]$$
$$=cx+(1-c)z .$$

Since λ is strictly convex fuzzy set , from the inequality (2) and equality above, we obtain

$$\lambda(z_1) > \lambda(x) \wedge \lambda(z) = \lambda(z)$$

contradicting (3).

(II) If $a<b<1$. That is $0<(b-a)/(1-a)<1$.

Let $z_2=[(b-a)/(1-a)]x+[(1-b)/(1-a)]y$,

thus ,

$$z=bx+(1-b)y=ax+(1-a)z_2 .$$

According to (A) , we have

$$\lambda(z) \geqslant \lambda(x) \wedge \lambda(z_2).$$

Again Since (2) and the inequality above imply

$$\lambda(z) \geqslant \lambda(z_2). \quad (4)$$

Let $d=(b-a)/[(1-a)b]$.

Since $0<a<b<1$, it is easy to show $0<d<1$.

Thus,

$$z_2=[1/(1-a)]z-[a/(1-a)]x$$
$$=[1/(1-a)]z-[a/(1-a)][\{1/b\}z-\{(1-b)/b\}y]$$
$$=dz+(1-d)y.$$

Since λ is strictly convex fuzzy set , from the inequality (2) and equality above, we obtain

$$\lambda(z_2) > \lambda(z) \wedge \lambda(y) = \lambda(z)$$

contradicting (4)

Theorem 2. Let λ be a convex fuzzy set on E . if there exists $a \in I$, for every $x, y \in E$, $\lambda(x) \neq \lambda(y)$ implies

$$\lambda(ax+(1-a)y) > \lambda(x) \wedge \lambda(y). \qquad (B)$$

Then λ is strictly convex fuzzy set on E.

Proof. By contradiction, suppose that there exist $x, y \in E$, $b \in I$ such that $\lambda(x) \neq \lambda(y)$ and

$$\lambda(bx+(1-b)y) \leqslant \lambda(x) \wedge \lambda(y). \qquad (5)$$

Without loss of generality , suppose that $\lambda(x) > \lambda(y)$.
Let $z = bx+(1-b)y$, then (5) implies

$$\lambda(z) \leqslant \lambda(x) \wedge \lambda(y) < \lambda(x). \qquad (6)$$

Since λ be convex fuzzy set , we have

$$\lambda(z) \geqslant \lambda(x) \wedge \lambda(y) \qquad (7)$$

which together with (5), we obtain

$$\lambda(x) > \lambda(z) = \lambda(x) \wedge \lambda(y) \qquad (8)$$

According to (B) , $\lambda(z) < \lambda(x)$ implies that

$$\lambda(ax+(1-a)z) > \lambda(x) \wedge \lambda(z) = \lambda(z),$$

$$\lambda(a^2x+(1-a^2)z) = f[a(ax+(1-a)z)+(1-a)z]$$

$$> \lambda[ax+(1-a)z] \wedge \lambda(z)$$

$$> \lambda(x) \wedge \lambda(z) = \lambda(z),$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\lambda[a^kx+(1-a^k)z] > \lambda(x) \wedge \lambda(z) = \lambda(z), \quad k \in N. \qquad (9)$$

From $z=bx+(1-b)y$, we have

$$a^kx+(1-a^k)z = a^kx+(1-a^k)(bx+(1-b)y)$$

$$= (b-a^kb+a^k)x+(1-b-a^k+a^kb)y$$

Let $\bar{k} \in N$ such that

$$a^{\bar{k}+1}/(1-a) < b/(1-b).$$

Let $e_1 = b-a^{\bar{k}}b+a^{\bar{k}}$, $e_2 = b-a^{\bar{k}+1}/(1-a)+a^{\bar{k}+1}b/(1-a)$,
$x^1 = e_1x+(1-e_1)y$, $y^1 = e_2x+(1-e_2)y$.

Thus ,

$$a^{\bar{k}}x+(1-a^{\bar{k}})z = e_1x+(1-e_1)y = x^1 , \qquad (10)$$

According to (9) and (10), we obtain

$$\lambda(x^1) = \lambda(e_1x+(1-e_1)y) = (a^{\bar{k}}x+(1-a^{\bar{k}})z) > \lambda(z) \qquad (11)$$

(Ⅰ) If $\lambda(x^1) \leqslant \lambda(y^1)$, according to $z=bx+(1-b)y=ax^1+(1-a)y^1$ and λ is convex fuzzy set , this implies that

$$\lambda(z) \geqslant \lambda(x^1) \wedge \lambda(y^1) = \lambda(x^1)$$

which contradicts (11).

(Ⅱ) If $\lambda(x^1) > \lambda(y^1)$, according to $z=ax^1+(1-a)y^1$ and (B), $\lambda(x^1) > \lambda(y^1)$ implies that

$$\lambda(z) > \lambda(x^1) \wedge \lambda(y^1) \qquad (12)$$

Again since $x^1=e_1x+(1-e_1)y$, $y^1=e_2x+(1-e_2)y$ and λ is convex fuzzy set, we have

$$\lambda(x^1) \geqslant \lambda(x) \wedge \lambda(y) . \tag{13}$$

$$\lambda(y^1) \geqslant \lambda(x) \wedge \lambda(y) . \tag{14}$$

According to (12), (13), and (14), we obtain

$$\begin{aligned} \lambda(z) &> \lambda(x^1) \wedge \lambda(y^1) \\ &\geqslant [\lambda(x) \wedge \lambda(y)] \wedge [\lambda(x) \wedge \lambda(y)] \\ &= \lambda(x) \wedge \lambda(y) \end{aligned}$$

which contradicts (8).

According to theorem 2 in [7] and theorem 2 above, we have the following corollary.

Corollary 3. Let λ be a fuzzy closed set on E, if there exists $a \in I$, for every $x, y \in E$, $\lambda(x) \neq \lambda(y)$ implies

$$\lambda(ax+(1-a)y) > \lambda(x) \wedge \lambda(y).$$

Then λ is a strictly convex fuzzy set on E. the converse is not true. But, we have the following result.

Theorem 4. Let λ be a convex fuzzy set on E, if there exists $a \in I$ such that for every pair of distinct points $x \in E$, $y \in E$ there holds

$$\lambda(ax+(1-a)y) > \lambda(x) \wedge \lambda(y). \tag{C}$$

Then λ is a strongly convex fuzzy set on E.

Proof. Assume that λ is not strongly convex fuzzy set on E. Then, there exist $x, y \in E$, $x \neq y$, $b \in I$ such that

$$\lambda(bx+(1-b)y) \leqslant \lambda(x) \wedge \lambda(y)$$

Let $z=bx+(1-b)y$, above inequality implies

$$\lambda(z) \leqslant \lambda(x) \wedge \lambda(y) . \tag{15}$$

Since λ is convex fuzzy set, we have

$$\lambda(z) \geqslant \lambda(x) \wedge \lambda(y) . \tag{16}$$

(15) and (16) imply

$$\lambda(z) = \lambda(x) \wedge \lambda(y) . \tag{17}$$

Choose β_1, β_2, satisfy $0<\beta_1<b<\beta_2<1$ such that

$$b=a\beta_1+(1-a)\beta_2 \tag{18}$$

Let $x^1=\beta_1x+(1-\beta_1)y$, $y^1=\beta_2x+(1-\beta_2)y$.

Thus,

$$ax^1+(1-a)y^1=bx+(1-b)y=z \tag{19}$$

Again since λ is convex fuzzy set, this implies

$$\lambda(x^1) \geqslant \lambda(x) \wedge \lambda(y) \tag{20}$$

$$\lambda(y^1) \geqslant \lambda(x) \wedge \lambda(y) \tag{21}$$

According to (C), (19), (20), and (21), we have

$$\begin{aligned} \lambda(z) &> \lambda(x^1) \wedge \lambda(y^1) \\ &\geqslant [\lambda(x) \wedge \lambda(y)] \wedge [\lambda(x) \wedge \lambda(y)] \\ &= \lambda(x) \wedge \lambda(y) , \end{aligned}$$

which contradicts (17).

According to corollary 1 in [7] and theorem 4 above, we have the following corollary.

Corollary 5. Let λ be a fuzzy closed set on E, if there exists $a\in I$, such that for every pair of distinct points $x\in E$, $y\in E$, implies that

$$\lambda(ax+(1-a)y) > \lambda(x)\wedge\lambda(y) .$$

Then λ be a strongly convex fuzzy set on E.

Theorem 6. Let λ be a strictly convex fuzzy set on E, if there exists $a\in I$, such that for every pair of distinct points $x\in E$, $y\in E$, implies that

$$\lambda(ax+(1-a)y) > \lambda(x)\wedge\lambda(y) . \qquad (D)$$

Then λ be a strongly convex fuzzy set on E.

Proof Since λ be a strictly convex fuzzy set, we only show $\lambda(x)=\lambda(y)$, $x\neq y$, implies that

$$\lambda(bx+(1-b)y) > \lambda(x)\wedge\lambda(y) , \qquad \vee\ b\in I .$$

Indeed, since (D), for each $x, y\in E$, $x\neq y$, we have

$$\lambda(ax+(1-a)y) > \lambda(x)\wedge\lambda(y)=\lambda(x)=\lambda(y)$$

let $x=ax+(1-a)y$. For each $b\in I$.

If $b<a$, we have

$$bx+(1-b)y=cx+(1-c)x , \quad \text{for some } c\in I .$$

Since λ is strictly convex fuzzy set on E, so

$$\lambda(bx+(1-b)y)=\lambda(cx+(1-c)x)$$
$$> \lambda(x)\wedge\lambda(x)=\lambda(x).$$

If $b>a$, we have

$$bx+(1-b)y=dx+(1-d)y , \quad \text{for some } d\in I .$$

Since λ is strictly convex fuzzy set on E, hence

$$\lambda(bx+(1-b)y)=\lambda(dx+(1-d)y)$$
$$> \lambda(x)\wedge\lambda(y)=\lambda(y).$$

This complete proof of theorem 6.

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