ON SOME MATHEMATICAL MODELS OF QUANTUM MECHANICAL SYSTEMS

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In this contribution we present a review of some mathematical models of quantum mechanical systems. A special attention is devoted to a fuzzy approach and consequently to some algebraic formulations including both fuzzy model as well as quantum logic model.

ORTHOMODULAR POSETS

The most famous model of quantum theory was due to J. v. Neumann [5]. He works with the family $L(H)$ of all linear subspaces of a given Hilbert space H. A state is a mapping $m\colon L(H) \to < 0,1 >$ satisfying the following two conditions:

(i) $m(H) = 1$

(ii) $m\left(\bigvee_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} m(a_i)$, whenever $a_i \subset a_j^{\perp}$ $(i \neq j)$.

(Here $\bigvee_{i=1}^{\infty} a_i$ is the subspace generated by the set union $\bigcup_{i=1}^{\infty} a_i$ of subspaces $a_1, a_2, \dots$ and $a_j^{\perp}$ is the orthogonal complement of the subspace a_j.)

An excellent and very useful algebraic generalization of the Hilbert space logic is the notion of an orthomodular poset [3]. It is a partially ordered set A with the greatest element 1 and the least element 0 and with a unary operation $a \mapsto a^{\perp}$ such that the following conditions are satisfied:

(i) If $(a_n)_{n=1}^{\infty} \subset A$ and $a_i \leqq a_j^{\perp} (i \neq j)$, then there exists the supremum $\bigvee_{i=1}^{\infty} a_i$.

(ii) $(a^{\perp})^{\perp} = a$, $a \vee a^{\perp}$ for every $a \in A$.

(iii) If $a \leq b$, then $b^{\perp} \leq a^{\perp}$ and $b = a \vee (a \wedge b^{\perp})$.

It is easy to see that every Hilbert space logic is an orthomodular poset, if the ordering is given by the inclusion and $a^{\perp}$ is the orthogonal complement of the subspace a. Moreover $L(H)$ is a lattice, where $a \wedge b$ is the set – theoretic intersection of a and b and $a \vee b$ is the subspace generated by a and b. Every orthomodular poset which is a lattice is called an orthomodular lattice.

Another example of an orthomodular poset is a q-σ-algebra ([3]) i.e. such a family Q of subsets of a set that $A \in Q$ implies $A^c \in Q$ and $\bigcup_{n=1}^{\infty} A_n \in Q$ whenever $A_n \in (n = 1, 2, \dots)$ and A_n are pairwise disjoint.

Evidently the notion of a state can be defined in every orthomodular poset by the following properties:

(i) $m(1) = 1$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

(ii) If $(a_i)_i \subset A$ and $a_i \leqq a_j^{\perp}$ $(i \neq j)$ then

$$m\Big(\bigvee_{i=1}^{\infty} a_i\Big) = \sum_{i=1}^{\infty} m(a_i).$$

Fuzzy quantum posets

The following model was introduced in [8] (see also [2]). It was inspired by the Piasecki P-measure [6]. It is a function $m\colon \mathcal{F} \to < 0,1 >$ defined on a family $\mathcal{F} \subset < 0,1 >^{\Omega}$ of fuzzy subsets of Ω and satisfying the following two conditions:

(i) If $f \in \mathcal{F}$, $f^{\perp} = 1 - f$ and $f \vee f^{\perp} = \max(f, f^{\perp})$, then $m(f \vee f^{\perp}) = 1$.
(ii) If $f_n \in \mathcal{F}$ $(n = 1, 2, \dots)$ and $f_n \leq f_m^{\perp} = 1 - f_m$ $(n \neq m)$, then

$$m\Big(\bigvee_{n=1}^{\infty} f_n\Big) = \sum_{n=1}^{\infty} m(f_n).$$

We see that there is an analogy between the P-measure on fuzzy ses and the state on an orthomodular poset. Therefore we introduced the following definitions.

An F-quantum space is a set $F \subset < 0,1 >^{\Omega}$ satisfying the following conditions:

(i) $1_{\Omega} \in \mathcal{F}$, $\left(\frac{1}{2}\right)_{\Omega} \notin \mathcal{F}$.
(ii) $f \in \mathcal{F} \Rightarrow 1 - f \in \mathcal{F}$.
(iii) $f_n \in \mathcal{F}$ $(n = 1, 2, \dots) \Rightarrow \bigvee_{n=1}^{\infty} f_n =: \sup_n f_n \in \mathcal{F}$.

If we use instead of (ii) a weaker condition

(iii') $f_n \in \mathcal{F}$ $(n = 1, 2, \dots)$, $f_n \leq 1 - f_m$ $(n \neq m) \Rightarrow \bigvee_{n=1}^{\infty} f_n \in \mathcal{F}$

then $\mathcal{F}$ is called F-quantum poset.

Evidently every q-σ-algebra Q can be undershood as an F-quantum poset, if we put

$$\mathcal{F} = \{\chi_A; A \in Q\}.$$

An F-state on $\mathcal{F}$ is a mapping $m\colon \mathcal{F} \to < 0,1 >$ satisfying the following two conditions:

(i) $m(f \vee f^{\perp}) = 1$ for every $f \in \mathcal{F}$.
(ii) If $f_n \in \mathcal{F}$ $(n = 1, 2, \dots)$, $f_n \leq f_m^{\perp}$ $(n \neq m)$, then

$$m\Big(\bigvee_{n=1}^{\infty} f_n\Big) = \sum_{n=1}^{\infty} m(f_n).$$

Orthocomplemented posets

Now we shall mention a common generalization of the preceeding concepts. A partially ordered set A is called an orthocomplemented poset ([4], WDO poset in [1]) if there is a unary operation $a \mapsto a^{\perp}$ satisfying the following conditions:

(i) $a \leq b \Rightarrow b^{\perp} \leq a^{\perp}$.
(ii) $(a^{\perp})^{\perp} = a$ for every $a \in A$.
(iii) If $(a_i)_i \subset A$, $a_i \leq a_j^{\perp} (i \neq j)$, then there exists $\bigvee_{i=1}^{\infty} a_i$ in A.

State is a mapping $m\colon A \to <0,1>$ satisfying the following conditions:

(i) $m(a \vee a^{\perp}) = 1$ for every $a \in A$.

(ii) $m\left(\bigvee_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} m(a_i)$ whenever $a_i \leq a_j^{\perp}$ $(i \neq j)$.

Similarly as states also observables can be defined in orthocomplemented posets. An observable is a mapping $x\colon \mathcal{B}(R) \to A(\mathcal{B}(R)$ in the family of all Borel subsets of R) satisfying the following conditions:

(i) $x(E') = x(E)^{\perp}$ for every $E \in \mathcal{B}(R)$.

(ii) $x\left(\bigcup_{n=1}^{\infty}\right) = \bigvee_{n=1}^{\infty} x(E_n)$ whenever $E_n \in \mathcal{B}(R)$ $(n = 1, 2, \dots)$.

If $m\colon A \to <0,1>$ is a state and $x\colon \mathcal{B}(R) \to A$ is an observable, then the composite map $m_x\colon \mathcal{B}(R) \to <0,1>$, defined by $m_x(E) = m\big(x(E)\big)$, is a probability measure. So some probability results can be achieved also in the general case.

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