ON MARTINGALES IN GENERAL ORDERED SYSTEMS

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In the contribution the martingale convergence theorem is proved in general ordered structure including the quantum logics as well as the F-quantum spaces.

1. QUASIORTHOCOMPLEMENTED POSETS

We shall assume that there is given a partially ordered set P with a mapping $a \mapsto a'$ from P to P satisfying the following conditions: a) If $a \leqq b$, then $b' \leqq a'$. b) $(a')' \geqq a$ for every $a \in P$. c) If $a_i \in P$ $(i = 1, 2, \dots)$ and $a_i \leqq a_{j'}$ $(i \neq j)$, then there is the supremum $\vee a_i$. The set P will be calleld quasi-orthocomplemented σ-poset.

Recall that this axiom system was used in [1], too.

Example 1.1. *Every quantum logic [2] satisfies the assumptions. A very special but important case is the logic P of all linear subspaces of a Hilbert space. P is ordered by inclusion and a' is the orthogonal complement of a.*

Example 1.2. *Every F-quantum space satisfies the assumptions stated above. F-quantum space is a set of functions $f\colon X \to \langle 0, 1\rangle$ containing 0, closed under complementation $f \mapsto 1 - f$ and countable unions $\bigvee_n f_n = \sup n f_n$. The notion of F-quantum space was inspired by Piasecki considerations [3] and it was introduced in [4] and [5]. Recently J. Pykacz [6] suggested from a physical point of view the following modification: F is closed under countably unions of pairwise orthogonal sequences $(f_n)_n$ (i. e. $f_n \leqq 1 - f_m$ for $n \neq m$). Evidently also this modification satisfies the above assumptions. (For recent development see [7].) Recall that every Gudder's $q - \sigma$-algebra of sets [8] is such a generalized F-quantum space, if we identify sets with their characteristic functions.*

Definition 1.1. *An observable is a σ-homomorphism x from the σ-algebra B of Borel subsets of the real line R to P, i.e. such a mapping $x\colon B \to P$ that a) $x(A') = x(A)'$ for every $A \in B$ and b) $x(\bigcup A_n) = \bigvee x(A_n)$ for every $A_n \in B$ $(n = 1, 2, \dots)$.*

If f is a random variable defined on a probability space (X, S, P), then the mapping $x\colon B \to S$ assigning to every $A \in B$ the pre-image $f^{-1}(A)$, is an observable. Also every observable in the quantum logic theory and every F-observable are observables in the sense of Definition 1.1.

Our results will be based on the following representation theorems ([1], [2]).

Theorem 1.1. *If Q is a countably generated sub-σ-algebra of P, then there is an observable y such that $Q = y(B)$.*

Theorem 1.2. *Let y, $z\colon B \to P$ be observables such that $x(B) \subset y(B)$. Then there exists a Borel measurable function $f\colon R \to R$ such that $z = y \circ f^{-1}$.*

Definition 1.2. *A state is a mapping $m\colon P \to \langle 0,1 \rangle$ satisfying the following two conditions: a) $m(a \vee a') = 1$ for every $a \in P$. b) $m(\vee a_n) = \sum m(a_n)$ for every $a_n \in P$ such that $a_n \leqq a'_m$ $(n \neq m)$.*

Proposition 1.1. *If $x\colon B \to P$ is an observable and $m\colon P \to \langle 0,1 \rangle$ is a state, then the mapping $m_x\colon B \to \langle 0,1 \rangle$ defined by $m_x(A) = m(x(A))$ is a probability measure.*

Definition 1.3. *An observable x is called integrable, if there exists $\int\limits_R t \, dm_x(t)$.*

The previous integral will be denoted by $\int x \, dm$. Our purpose is to define the indefinite integral $\int\limits_a x \, dm$.

Conditional expectation

Definition 2.1. *Let x, y be observables, $x(B) \subset y(B) = Q, a \in Q$. Then we define $x_a\colon B \to P$ by the formula $x_a(U) = y(T^{-1}(h^{-1})))$, where $h\colon R_2 \to R$ is defined by $h(u,v) = uv$ and $T\colon R \to R_2$ is defined by $T(u) = (f(u),\ 1_A(u))$, $x = y \circ f^{-1}$ and $a = y(A)$.*

As a motivation we shall consider the classical case, where y is induced by a random variable g, hence $y = g^{-1}$, $x = g^{-1} \circ f^{-1} = k^{-1}$, $k = f \circ g$.

Then $(k \cdot 1_a)^{-1}(U) = \big((f \circ g) \cdot (1_A \circ g)\big)^{-1}(U) = \big((f \cdot 1_A) \circ g\big)^{-1}(U) = (h \circ T \circ g)^{-1}(U) = g^{-1} \circ T^{-1} \circ (h^{-1}(U) = y(T^{-1}(h^{-1}(U)))$.

Of course, we must prove the corectness of Definition 2.1.

Lemma 2.1. *Let x, y, $z\colon B \to P$ be observables and f, $g\colon R \to R$ be Borel functions such that $x = y \circ f^{-1} = z \circ g^{-1}$. Let $a \in y(B) = z(B)$ and let $a = y(A) = x(C)$. Put $T\colon R_2 \to R$, $S\colon R_2 \to R$ by $T(u) = (f(u),\ 1_A(u))$, $S(u) = (g(u),\ 1_C(u))$. Then $y \circ T^{-1} \circ h^{-1} = z \circ S^{-1} \circ h^{-1}$.*

Proof. Put first $D = h^{-1}(U)$ and assume that $D = E \times F$, where E, $F \in B$. then

$$y(T^{-1}(h^{-1}(U))) = y(T^{-1}(D)) = y(T^{-1}(E \times F) = y(f^{-1}(E) \circ 1_A^{-1}(F)) = \\ = y(f^{-1}(E)) \wedge y(1_A^{-1}(F)).$$

But $y(f^{-1}(E)) = x(E) = z(g^{-1}(E))$ by the assumption. We prove that also $y(1_A^{-1}(F)) = z(1_C^{-1}(F))$. If e. g. $0 \notin F$ and $1 \in F$, then $1_A^{-1}(F) = A$, $1_C^{-1}(F) = C$, hence $y(1_A^{-1}(F)) = y(A) = a = z(C) = z(1_C^{-1}(F))$; similarly also all other cases can be examined. So

$$y(T^{-1}(h^{-1}(U))) = y(f^{-1}(E)) \wedge y(1_A^{-1}(F)) = \\ = z(g^{-1}(E)) \wedge z(1_C^{-1}(F)) = z(S^{-1}(h^{-1}(U))).$$

Since the family $K = \{U \in B_2;\ y(T^{-1}(h^{-1}(U))) = z(S^{-1}(h^{-1}(U)))\}$ is a σ-algebra containing the family $\{E \times F;\ E \in B,\ F \in B\}$, $K \supset B_2$.

Theorem 2.1. *If x is integrable, then x_a is integrable, too. Moreover, if we define $\int_a x\,dm = \int x_a$, then $\int_a x\,dm = \int_A f\,dm_y$.*

Proof. By the definition and the integral transformation formula (i is the identity map)

$$\int_R i\,dm_x = \int_{f^{-1}(R)} i\,dm_y \circ f^{-1} = \int_R f\,dm_y\,.$$

Therefore f is m_y-integrable, hence $f \cdot 1_A$ is m_y-integrable, too.
Moreover

$$\int_A f\,dm_y = \int_R f \cdot 1_A\,dm_y = \int_R i\,dm_y \circ (f \cdot 1_A)^{-1}\,.$$

But

$$y \circ (f \cdot 1_A)^{-1} = y \circ \big(h(f, 1_A)\big)^{-1} = y \circ T^{-1} \circ h^{-1} = x_a\,,$$

hence

$$m_y \circ (f \cdot 1_A)^{-1} = m \circ y \circ (f \cdot 1_A)^{-1} = m(x_a) = \int_a x\,dm\,.$$

Theorem 2.2. *Let Q_0, $Q \subset P$ be Boolean sub-σ-algebras of P, $Q_0 \subset Q$. Let $x\colon B \to P$ be an observable such that $x(B) \subset Q$. Then there exists an observable $z\colon B \to P$ such that $z(B) \subset Q_0$ and*

$$\int_a z\,dm = \int_a x\,dm$$

for every $a \in Q_0$.

Proof. By Theorem 2.1 there is an m_y-integrable function $f\colon R \to R$ such that

$$\int_A f\,dm_y = \int_a x\,dm$$

for every $a \in Q$ and every $A \in B$ such that $y(A) = a$. Put $S_0 = \{E \in B;\, y(E) \in Q_0\}$. Then S_0 is a sub-σ-algebra of B. Put $g = E(f \,|\, S_0)$. Then g is S_0-measurable and

$$\int_A f\,dm_y = \int_A g\,dm_y$$

for every $A \in S_0$. Put $z = y \circ g^{-1}$. Then $z(B) = z\big(g^{-1}(B)\big) \subset z(S_0) = Q_0$. Futher, if $a \in Q_0$, then $A \in S_0$, hence

$$\int_a x\,dm = \int_A f\,dm_y = \int_A g\,dm_y\,.$$

By the integral transformation formula

$$\int\limits_R g\ dm_y = \int\limits_R i\ dm_y \circ g^{-1} = \int\limits_R i\ dm_z\,,$$

since $m_z(E) = m\big(z(E)\big) = m\big(x(g^{-1}(E))\big) = m_y\big(g^{-1}(E)\big)$. Hence z is integrable, $z = y \circ g^{-1}$. Therefore by Theorem 2.1

$$\int\limits_a z\ dm = \int\limits_A g\ dm_y\,.$$

3. Martingale convergence theorem

The classical convergence theorem says about the convergence almost everywhere. Therefore we must first define this kind of convergence of observables in the general case.

Definition 3.1. *We say that a sequence (x_n) of observables converges to 0 m-almost everywhere, if*

$$m\left(\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty} x_n(R\setminus(-\varepsilon,\varepsilon))\right) = 0$$

for every $\varepsilon > 0$.

Generally (x_n) converges to x, if $(x_n - x)$ converges to 0. Of course, it is necessary to define the difference of observables.
Again we shall follow the classical example. Let x be an observable induced by a Borel function f, i.e. $x = f^{-1}$, y be induced by g, i. e. $y = g^{-1}$. Put $h\colon R_2 \to R$, $h(u,v) = u - v$, $T\colon X \to R_2$, $T(u) = \big(f(u), g(u)\big)$. Then

$$(f-g)^{-1} = \big(h(f,g)\big)^{-1} = (h\circ T)^{-1} = T^{-1}\circ h^{-1}\,.$$

Here $T^{-1}\colon B_2 \to S$ is such a σ-homomorphism, that $T^{-1}(E\times F) = f^{-1}(E)\wedge g^{-1}(F)$. So we can generalize.

Definition 3.2. *Let x, y be observables. We shall say that there exists the difference $z = x - y$, if there is a σ-homomorphism $k\colon B_2 \to P$ such that $k(E\times F) = x(E)\wedge y(F)$. Then we define $x - y(C) = k\big(h^{-1}(C)\big)$, where $h\colon R_2 \to R$ is defined by $h(u,v) = u - v$.*

Definition 3.3. *We say that a sequence (x_n) of observables converges to an observable x m-almost everywhere, if $x_n - x$ there exists for every n (sufficiently large) and $(x_n - x)$ converges to 0 m-a.e.*

Theorem 3.1. *If z_1, z_2 are two observables satisfying the conditions stated in Theorem 2.2, then $z_1 = z_2$ m-a.e., i.e. $m\big((z_1 - z_2)(\{0\})\big) = 0$.*

Proof. Let $z_1 = y \circ g_1^{-1}$, $z_2 = y \circ g_2^{-1}$. Then $k = y \circ (g_1, g_2)^{-1}$ satisfies the condition stated in Definition 3.2, hence $z_1 - z_2 = k \circ h^{-1} = y \circ (g_1, g_2)^{-1} \circ h^{-1} = y \circ \big(h \circ (g_1, g_2)\big)^{-1} = y \circ (g_1 - g_2)^{-1}$. Further $g_1 = g_2$ m_y-a.e., since g_1, g_2 are variants of the same conditional expectation $E(f \mid S_0)$. Therefore

$$0 = m_y\big(\{u;\, g_1(u) = g_2(u)\}\big) = m\big(y((g_1 - g_2)^{-1}(\{0\}'))\big) = \\ = m\big(z_1 - z_2(\{0\})')\big)\,.$$

Theorem 3.2. *Let (Q_n) be a sequence of sub-σ-algebras of P such that $Q_n \subset Q_{n+1} \subset Q$ $(n = 1, 2, \dots)$ and Q is countably generated. Let (x_n) be a sequence of integrable observables such that $x_n(B) \subset Q_n$ $(n = 1, 2, \dots)$. Assume that $E(x_{n+1} \mid Q_n) \geqq x_n$ for every n and that $\sup\limits_n \int x_n \, dm < \infty$. Then there exists an observable x such that $x_n \to x$ m-a.e.*

Proof. Let y be an observable such that $Q = y(B)$. Let f_n be Borel measurable functions such that $x_n = y \circ f_n^{-1}$. Finally let $S_n = \{E \in B;\, y(E) \in Q_n\}$. Then $\big((f_n, S_n)\big)$ is submartingale with $\sup \int f_n \, dm_y \leq \infty$. Therefore there is an integrable random variable f (defined on the probability space (R, B, m_y)) such that $f_n \to f$ m_y-a.e. Put $x = y \circ f^{-1}$. Then $x_n \to x$ m-a.e.

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