Carathéodory's Measurability of Fuzzy Events

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Abstract. For a given fuzzy measurable space $(\mathbf{X},\sigma)$ and a fuzzy set function $\mathbf{m}:\sigma\longrightarrow\mathbf{R}$ we define the system $\sigma_{\mathbf{m}}$ of all fuzzy sets measurable in the sense of Carathéodory. We study the properties of $\sigma_{\mathbf{m}}$. Under simple general conditions we show that $\sigma_{\mathbf{m}}$ is a soft fuzzy algebra and $\mathbf{m}$ is a fuzzy P-measure on $\sigma_{\mathbf{m}}$.

Let $\mathbf{X}$ be a nonempty set. The symbol $\mathcal{F}(\mathbf{X})$ denotes the family of all fuzzy subsets of $\mathbf{X}$, i.e. $\mathcal{F}(\mathbf{X}) = [0,1]^{\mathbf{X}}$. The operations of the fuzzy union, fuzzy intersection and fuzzy complement are defined in the traditional Zadeh's sense [5],

$(A \cup B)(x) = \sup\ (A(x),B(x))$

$(A \cap B)(x) = \inf\ (A(x),B(x))$

$A'(x) = 1 - A(x)$

for any $A,\ B \in \mathcal{F}(\mathbf{X})$ and $x \in \mathbf{X}$.

A family $\sigma \subset \mathcal{F}(\mathbf{X})$ of fuzzy subsets of $\mathbf{X}$ is called a *fuzzy algebra* (*fuzzy σ-algebra*), see [2], iff:

(A1) $0_{\mathbf{X}} \in \sigma$ (we put $c_{\mathbf{X}}(x) = c$ for any $x \in \mathbf{X}$, $c \in [0,1]$)

(A2) $A \in \sigma$ implies $A' \in \sigma$

(A3) $A,\ B \in \sigma$ implies $A \cup B \in \sigma$ ($\{A_n\}_{n\in\mathbf{N}}$ implies $\bigcup_{n\in\mathbf{N}} A_n \in \sigma$ in the case of a fuzzy σ-algebra).

Let $\mathbf{m}$ be a function defined on the fuzzy sets from σ, $\mathbf{m}:\sigma\longrightarrow\mathbf{R}$, where $\mathbf{R}$ is the set of all real numbers. We will distinguish a family $\sigma_{\mathbf{m}} \subset \sigma$ of fuzzy sets measurable in the Carathéodory's sense following the classical concept introduced in [1],

$$\sigma_{\mathbf{m}} = \left\{ A \in \sigma;\ \forall\ B \in \sigma:\ \mathbf{m}(B) = \mathbf{m}(B \cap A) + \mathbf{m}(B \cap A') \right\}$$

For the above defined system $\sigma_{\mathbf{m}}$ we have the following results.

Proposition 1.

i) $\sigma_{\mathbf{m}}$ is closed under complementation, i.e. $A \in \sigma_{\mathbf{m}} \Longrightarrow A' \in \sigma_{\mathbf{m}}$

ii) $\sigma_{\mathbf{m}}$ is nonempty iff $\mathbf{m}(0_X) = 0$ iff $\{0_X; 1_X\} \subset \sigma_{\mathbf{m}}$

iii) $A \in \sigma_{\mathbf{m}} \Longrightarrow \mathbf{m}(A \cap A') = 0$ and $\mathbf{m}(A \cup A') = \mathbf{m}(A) + \mathbf{m}(A') = \mathbf{m}(1_X)$

iv) Let $(1/2)_X \in \sigma$; then $(1/2)_X \in \sigma_{\mathbf{m}}$ iff $\mathbf{m}(A) = 0$ for all $A \in \sigma$.

Proof.

i) is obvious because of $A'' = A$.

ii) Let $\sigma_{\mathbf{m}} \neq \emptyset$. Then there is an element $A \in \sigma_{\mathbf{m}}$ and hence $\mathbf{m}(0_X) = \mathbf{m}(0_X \cap A) + \mathbf{m}(0_X \cap A') = 2\mathbf{m}(0_X)$ and consequently $\mathbf{m}(0_X) = 0$. Now, let $\mathbf{m}(0_X) = 0$. For any $A \in \sigma$ it is $\mathbf{m}(A) = \mathbf{m}(A \cap 0_X) + \mathbf{m}(A \cap 1_X)$ and hence both 0_X and 1_X are contained in $\sigma_{\mathbf{m}}$. The last implication, namely that $\{0_X; 1_X\} \subset \sigma_{\mathbf{m}}$ implies that $\sigma_{\mathbf{m}}$ is nonempty, is obvious.

iii) Let $A \in \sigma_{\mathbf{m}}$. Then

$\mathbf{m}(A \cap A') = \mathbf{m}(A \cap A' \cap A) + \mathbf{m}(A \cap A' \cap A') = 2\mathbf{m}(A \cap A') = 0$. Further

$$\mathbf{m}(A \cup A') = \mathbf{m}((A \cup A') \cap A) = \mathbf{m}((A \cup A') \cap A') = \mathbf{m}(A) + \mathbf{m}(A') =$$
$$= \mathbf{m}(1_X \cap A) + \mathbf{m}(1_X \cap A') = \mathbf{m}(1_X) .$$

iv) Let $(1/2)_X \in \sigma_{\mathbf{m}}$. Then $(1/2)_X' = (1/2)_X$ and for any $A \in \sigma$ it is

$$\mathbf{m}(A) = \mathbf{m}(A \cap (1/2)_X) + \mathbf{m}(A \cap (1/2)_X') = 2\mathbf{m}(A \cap (1/2)_X) =$$
$$= 2.2\mathbf{m}(A \cap (1/2)_X)$$ and consequently $\mathbf{m}(A) = 0$.

On the other hand, if $\mathbf{m}(A) = 0$ for any $A \in \sigma$, then $\sigma_{\mathbf{m}} = \sigma$ and hence $(1/2)_X \in \sigma$ implies $(1/2)_X \in \sigma_{\mathbf{m}}$. ∎

To ensure $\sigma_{\mathbf{m}}$ to be a fuzzy algebra, it is enough to ensure $\sigma_{\mathbf{m}}$ to be nonempty and closed under fuzzy union (or fuzzy intersection).

Proposition 2. Let $\mathbf{m}(0_X) = 0$ and let $\mathbf{m}$ satisfies the null-additivity condition

(*) $N \in \sigma$, $\mathbf{m}(N) = 0 \Longrightarrow \forall A \in \sigma$ it is $\mathbf{m}(A) = \mathbf{m}(A \cup N)$.

Then $\sigma_{\mathbf{m}}$ is a fuzzy algebra.

Proof.

Let $A, B \in \sigma_{\mathbf{m}}$. Then for any $C \in \sigma$ we have

$$\mathbf{m}(C) = \mathbf{m}(C \cap A) + \mathbf{m}(C \cap A') = \mathbf{m}(C \cap A \cap B) + \mathbf{m}(C \cap A \cap B') + \mathbf{m}(C \cap A')$$

On the other hand, it is

$$\mathbf{m}(C \cap (A \cap B)') = \mathbf{m}((C \cap A') \cup (C \cap B')) =$$
$$= \mathbf{m}((C \cap A' \cap A) \cup (C \cap B' \cap A)) + \mathbf{m}((C \cap A' \cap A') \cup (C \cap B' \cap A')) =$$
$$= \mathbf{m}(C \cap B' \cap A) + \mathbf{m}(C \cap A') .$$

The previous equality follows from $\mathbf{m}(A' \cap A) = 0$ and the null-additivity of $\mathbf{m}$. Note that the null-additivity implies $\mathbf{m}(H) = 0$ for any fuzzy subset $H \in \sigma$ such that $H \subset N$ for some N fulfilling $\mathbf{m}(N) = 0$.

Now, it is evident that

$$\mathbf{m}(C) = \mathbf{m}(C \cap (A \cap B)) + \mathbf{m}(C \cap (A \cap B)')$$

and hence the fuzzy intersection $(A \cap B)$ is an element of $\sigma_{\mathbf{m}}$. Thus $\sigma_{\mathbf{m}}$ is a fuzzy algebra. ∎

Note that the null-additivity (*) is fulfilled e.g. for subadditive fuzzy measures. If the condition (*) is not fulfilled, $\sigma_{\mathbf{m}}$ may be not closed under union and intersection.

<u>Example 1.</u> Let $\mathbf{X} = \{x_1, x_2\}$, $\sigma = \mathcal{F}(\mathbf{X}) = \left\{A = (a_1, a_2) \in [0,1]^2\right\}$, and let $\mathbf{m}(A) = \min\{2a_1, 2a_2, 1\}$. Then

1) $\mathbf{m}(0_{\mathbf{X}}) = 0$ and hence $\sigma_{\mathbf{m}}$ is nonempty
2) $\mathbf{m}(1_{\mathbf{X}}) = 1$ and hence $(1/2)_{\mathbf{X}}$ is not contained in $\sigma_{\mathbf{m}}$
3) Let $A = (1/2, 0)$. Then for any $B \in \sigma$ it is

$$\mathbf{m}(A \cap B) = \min\left\{\min\{1, 2b_1\}, \min\{0, 2b_2\}, 1\right\} = 0 \quad ,$$

$$\mathbf{m}(A' \cap B) = \min\left\{\min\{1, 2b_1\}, \min\{2, 2b_2\}, 1\right\} =$$

$$= \min\{2b_1, 2b_2, 1\} = \mathbf{m}(B) \text{ , so that}$$

$m(B) = \mathbf{m}(B \cap A) + \mathbf{m}(B \cap A')$. This implies $A \in \sigma_{\mathbf{m}}$. Similarly we can show that $C = (0, 1/2) \in \sigma_{\mathbf{m}}$. But then $A \cup C = (1/2)_{\mathbf{X}}$, so that $\sigma_{\mathbf{m}}$ is not closed under fuzzy union. ∎

<u>Proposition 3.</u> Let $\{A_1, \ldots, A_k\} \subset \sigma_{\mathbf{m}}$, $k \in \mathbf{N}$, be a system of pairwise W-disjoint fuzzy subsets (see e.g. [4]), i.e. $A_i \le A_j'$ whenever $i \neq j$. Then under the condition of Proposition 2 it is

$$\mathbf{m}\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} \mathbf{m}(A_i) \quad .$$

<u>Proof.</u>

Let $k = 2$, i.e. $A_1 = A$, $A_2 = B$ and $A \le B'$, $A, B \in \sigma_{\mathbf{m}}$. Then $\mathbf{m}(A \cap A')$ is zero and $(A \cap B) \le (A \cap A')$, so that $\mathbf{m}(A \cap B) = 0$. Then the null additivity of $\mathbf{m}$ implies

$\mathbf{m}(A \cup B) = \mathbf{m}((A \cup B) \cap A) + \mathbf{m}((A \cup B) \cap A') = \mathbf{m}(A) + \mathbf{m}(B \cap A')$ and

$\mathbf{m}(B) = \mathbf{m}(B \cap A) + \mathbf{m}(B \cap A') = \mathbf{m}(B \cap A')$. It follows

$\mathbf{m}(A \cup B) = \mathbf{m}(A) + \mathbf{m}(B)$. For $k \ge 2$ we can use the induction because of the pairwise W-disjointness of $\{A_1, \ldots, A_k\}$ implies the W-disjointness of $(A_1 \cup \ldots \cup A_{k-1})$ and A_k . ∎

Definition (see [3]). Let σ be a soft (i.e. $(1/2)_X \notin \sigma$) fuzzy algebra (fuzzy σ-algebra). A mapping $\mathbf{p}:\sigma \longrightarrow [0,1]$ is called a fuzzy P-measure on σ if the followings are fulfilled:

i) $\forall\ A \in \sigma : \mathbf{p}(A \cup A') = 1$

ii) $\forall\ \{A_n\} \subset \sigma$, $A_n \le A_m'$ whenever $n \ne m$, and $(\bigcup A_n) \in \sigma$:

$\mathbf{p}(\bigcup A_n) = \sum \mathbf{p}(A_n)$. ■

Corollary 1. Let $\mathbf{m}(0_X) = 0$, $\mathbf{m}(1_X) = 1$ and let $\mathbf{m}$ be null-additive on σ. Then $\sigma_\mathbf{m}$ is a soft fuzzy algebra and if $\mathbf{m}$ is continuous from below, then $\mathbf{m}$ restricted to $\sigma_\mathbf{m}$ is a fuzzy P-measure on $\sigma_\mathbf{m}$.

Proof.

It is enough to prove the validity of the condition ii) for $\mathbf{m}/\sigma_\mathbf{m}$. For finite sequences the result is contained in Proposition 3. For infinite sequences $\{A_n\}$, the continuity from below of $\mathbf{m}$ implies

$\mathbf{m}(\bigcup A_n) = \lim_k \mathbf{m}(A_1 \cup \ldots \cup A_k) = \lim_k (\mathbf{m}(A_1) + \ldots + \mathbf{m}(A_k)) = \sum \mathbf{m}(A_n)$. ■

Note that if $\mathbf{m}$ is continuous on a fuzzy σ-algebra σ , then if $\sigma_\mathbf{m}$ is a soft fuzzy algebra, it is also a soft fuzzy σ-algebra. However, this condition is not a necessary one.

Example 2. We take X and σ from Example 1. We put

$\mathbf{m}(A) = (1_{]1/4,1]}(a_1) + 1_{]1/4,1]}(a_2))/2$ for any $A \in \sigma$.

Then $\mathbf{m}$ is not continuous on σ, as $(1/4)_X = \lim_n (1/4 + 1/n)_X$, but

$\mathbf{m}((1/4)_X) = 0 \ne 1 = \lim_n \mathbf{m}((1/4 + 1/n)_X)$. On the other hand,

$$\sigma_\mathbf{m} = \left\{A \in \sigma,\ a_1 \notin\ ]1/4,3/4[\right\}$$

is a soft fuzzy σ-algebra. ■

Proposition 4. Let $(X,\sigma,\mathbf{p})$ be a given fuzzy probability space, i.e. σ is a soft fuzzy algebra and $\mathbf{p}$ is a fuzzy P-measure on σ. Then $\sigma_\mathbf{p} = \sigma$.

Proof.

Let $A \in \sigma$. For any $B \in \sigma$ it is $(B \cap A) \le (B \cap A')' = (B \cup A')$ and hence $(B \cap A)$ and $(B \cap A')$ are W-disjoint elements of σ. Then

$\mathbf{p}(B \cap A) + \mathbf{p}(B \cap A') = \mathbf{p}((B \cap A) \cup (B \cap A')) = \mathbf{p}(B \cup (A \cap A')) = \mathbf{p}(B)$,

and consequently $A \in \sigma_\mathbf{m}$. For the last equality see e.g. [4]. ■

Remark. Proposition 4 shows that under the conditions of Corollary 1, $\sigma_{\mathbf{m}}$ is the greatest soft fuzzy subalgebra of σ on which $\mathbf{m}$ is a fuzzy P-measure. ∎

The assertion iii) of Proposition 1 gives a necessary condition for an element A of σ to be an element of $\sigma_{\mathbf{m}}$, namely $\mathbf{m}(A \cap A') = 0$ and $\mathbf{m}(A \cup A') = \mathbf{m}(1_X)$. However, this condition is not sufficient.

Example 3. Again we take X and σ from Example 1. Now, we define

$$\mathbf{m}(A) = max \left\{(2max \{a_1, a_2\} -1), 0\right\} \text{ for any } A \in \sigma .$$

$\mathbf{m}$ is null-additive and $\sigma_{\mathbf{m}}$ is a soft fuzzy σ-algebra. Take A = (1,1/4). Then $\mathbf{m}(A \cap A') = \mathbf{m}((0,1/4)) = 0$ and $\mathbf{m}(A \cup A') = \mathbf{m}((1,3/4)) = 1$, but $\mathbf{m}(1_X) = 1 \neq 3/2 = \mathbf{m}(A) + \mathbf{m}(A') = \mathbf{m}(1_X \cap A) + \mathbf{m}(1_X \cap A')$. It follows that A is not an element of $\sigma_{\mathbf{m}}$. ∎

In the following proposition we give the conditions which make the previous necessary condition for an element of $\sigma_{\mathbf{m}}$ a sufficient one.

Proposition 5. Let $\mathbf{m}$ be a monotone valuation on σ, i.e. for $A \subset B$ it is $\mathbf{m}(A) \le \mathbf{m}(B)$ and $\mathbf{m}(A) + \mathbf{m}(B) = \mathbf{m}(A \cap B) + \mathbf{m}(A \cup B)$ for any A, B $\in \sigma$. Then

$$\sigma_{\mathbf{m}} = \left\{A \in \sigma;\ \mathbf{m}(A \cap A') = 0 \text{ and } \mathbf{m}(A \cup A') = \mathbf{m}(1_X)\right\} .$$

Proof.

It is enough to prove that $\mathbf{m}(A \cap A') = 0$ and $\mathbf{m}(A \cup A') = \mathbf{m}(1_X)$ for an element $A \in \sigma$ implies $A \in \sigma_{\mathbf{m}}$. Recall that $\sigma_{\mathbf{m}}$ is nonempty iff $\mathbf{m}(0_X)$ equals zero. Take any $B \in \sigma$. Then the valuation property of $\mathbf{m}$ implies

$$\mathbf{m}(B \cap A) + \mathbf{m}(B \cap A') = \mathbf{m}((B \cap A) \cup (B \cap A')) + \mathbf{m}((B \cap A) \cap (B \cap A')) =$$
$$= \mathbf{m}(B \cap (A \cup A')) + \mathbf{m}(B \cap (A \cap A')) .$$

From the monotonicity of $\mathbf{m}$ it is $0 \le \mathbf{m}(B \cap (A \cap A')) \le \mathbf{m}(A \cap A') = 0$, i.e. $\mathbf{m}(B \cap (A \cap A')) = 0$. Further, the valuation property implies

$$\mathbf{m}(B \cap (A \cup A')) = \mathbf{m}(B) + \mathbf{m}(A \cup A') - \mathbf{m}(B \cup (A \cup A')) = 1$$

because of $1 = \mathbf{m}(A \cup A') \le \mathbf{m}(B \cup (A \cup A')) \le 1$. It follows $\mathbf{m}(B) = \mathbf{m}(B \cap A) + \mathbf{m}(B \cap A')$, so that $A \in \sigma_{\mathbf{m}}$. ∎

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