ON FUZZY EVENTS ILL DEFINED UNDER KLEMENT"S FUZZY PROBABILITY MEASURE

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1. INTRODUCTION

The notion of ill defined fuzzy events is tentatively proposed in [5] where it follows from some considerations on the Bayes principle for fuzzy events. The main goal of this paper is a presentation of the final version of ill defined fuzzy random events and investigation of some properties of redefined here notion. First, recall the motivation on which is based the definition of ill defined fuzzy events.

Let (X,σ) be a given fuzzy measurable space. Here X is a nonempty crisp set of elementary events and σ is a family of fuzzy random events defined as a fuzzy algebra in the sense given by Klement et al. [2] , i.e. σ is closed under complementation and union, and it contains the fuzzy subset $0_X : X \longrightarrow \{0\}$.
Note, that union and intersection of fuzzy subsets are expressed here respectively by Zadeh's operators supremum $\vee$ and infimum $\wedge$. The operator of complementation is given here as the Lukasiewicz's negation [4] , $\mu' = 1 - \mu$. By $\mathcal{F}(X)$ we denote the family of all fuzzy subsets of X, i.e. $\mathcal{F}(X) = [0,1]^X$. Start-point of our considerations is based on next definitions:

<u>Definition</u> <u>1.1.</u>[7]: Each pair of fuzzy subsets $\mu, \nu \in \sigma$ is called W-disjoint iff the first is contained in the complement of the second, i.e. $\mu \le \nu'$.

<u>Definition</u> <u>1.2.</u>[7]: Each fuzzy subset $\mu \in \sigma$ is called W-empty set iff it is W-disjoint with itself, i.e. iff it is contained in its complement, $\mu \le \mu'$.

<u>Definition</u> <u>1.3.</u>[7]: Each fuzzy subset $\mu \in \sigma$ is called W-universum iff its complement is W-empty set, i.e. iff it contains its

complement, $\mu' \le \mu$.

In the crisp case, the classical "sound" empty set A fulfills following three conditions: A is an empty set, the complement A' is an universum, $A' \neq A$. For the fuzzy case we denote the set of all "sound" W-empty sets from σ by the symbol $\varepsilon(\sigma)$ describing the next family

$$\varepsilon(\sigma) = \{ \mu \in \sigma: \mu \text{ is W-empty set}, \mu' \text{ is W-universum}, \mu \neq \mu' \}.$$

The classical measure theory essentially exploits the fact that for each crisp subset A, $A \cap A'$ is a "sound" empty set. There is no "ill" defined element A, for which $A \cap A' \neq 0$. In the fuzzy case, we are in a quite different situation. We will denote the set of all W-ill defined elements connected with fuzzy algebra σ by the symbol $\rho(\sigma)$ describing the following family

$$\rho(\sigma) = \{ \mu \in F(X): \mu \wedge \mu' \notin \varepsilon(\sigma) \}.$$

It is easy to see that for any σ, $\rho(\sigma) \ni \left[\frac{1}{2} \right]_X$. Thus note that if the fuzzy algebra σ does not contain any W -ill defined element, i.e. $\sigma \cap \rho(\sigma) = \emptyset$, then σ is a soft fuzzy algebra distinguished by Piasecki [8] (a fuzzy algebra not containing the element $\left[\frac{1}{2} \right]_X$. Moreover, in this case we can define a probability of fuzzy random events from σ as a fuzzy P -measure described by Piasecki in following manner.

Definition 1.4. [8] : Let $\sigma \subset \mathcal{F}(X)$ be a soft fuzzy algebra. Each mapping p: $\sigma \longrightarrow [0,1]$ satisfying next conditions:

(P1) if $\mu \in \sigma$ is a W -universum then $p(\mu) = 1$

(P2) if $\{\mu_n\} \subset \sigma$ is such sequence of W -disjoint fuzzy random events ($\mu_i \le \mu_j'$ whenever $i \neq j$) that $\sup_n\{\mu_n\} \in \sigma$ then

$$p(\sup_n\{\mu_n\}) = \sum_n p(\mu_n) \quad ;$$

is called a fuzzy P -measure on σ.

On the other side, in [5] the relation of W -disjointness is replaced by the relation of disjointness under measure (m -disjointness). Some consequences of this replacement will be investigated bellow.

2. m-ILL DEFINED ELEMENTS

Let m: $\sigma \longrightarrow [0,1]$ be a fuzzy probability measure in the sense of Klement et al. [2], i.e. the mapping m satisfies following conditions:

$$m(0_X) = 0 \quad (2.1.) \qquad ; \qquad m(1_X) = 1 \quad (2.2)$$

$$\forall\ \mu,\nu \in \sigma :\ m(\mu) + m(\nu) = m(\mu \vee \nu) + m(\mu \wedge \nu) \qquad (2.3)$$

$$\forall\ \{\mu_n\} \subset \sigma :\ \mu_n \nearrow \mu \in \sigma \Rightarrow m(\mu_n) \nearrow m(\mu) \qquad (2.4)$$

In [5] we have introduced the concept of disjointness under measure . There was proposed :

Definition 2.1. : Each pair of fuzzy subsets $\mu,\nu \in \sigma$ is called m -disjointness iff $m(\mu \wedge \nu) = 0$.

The disjointness implies the definition of an empty set. In the crisp case, a set A is empty iff it is disjoint with itself. Therefore, using m -disjointness we get $m(\mu) = 0$ for fuzzy subset μ which is empty under measure.

Definition 2.2. : Each fuzzy subset $\mu \in \sigma$ is called m -empty set iff $m(\mu) = 0$.

The notion of an universum is a dual notion to that of an empty set.

Definition 2.3. : Each fuzzy subset $\mu \in \sigma$ is called m -universum iff $m(\mu) = 1$.

The family $\varepsilon_m(\sigma)$ of all "sound" m -empty subset from σ we distinguish in like way as the family $\varepsilon(\sigma)$ of all "sound" W -empty subset from σ. We propose to accept

$$\varepsilon_m(\sigma) = \{\ \mu \in \sigma : \mu \text{ is m -empty set, } \mu' \text{ is m -universum}\} \qquad (2.5)$$

Note that for all $\mu \in \varepsilon_m(\sigma)$ we have $\mu \neq \mu'$. Therefore, the third condition from the definition of $\varepsilon(\sigma)$ is omitted above.

The family $\rho_m(\sigma)$ of all m -ill defined elements is as follows

$$\rho_m(\sigma) = \{\ \mu \in F(X):\ \mu \wedge \mu' \notin \varepsilon_m(\sigma)\} \qquad (2.6)$$

Moreover, in [6] the family σ_m of measurable fuzzy random events from σ is distinguished in the following manner

$$\sigma_m = \{\ \mu \in \sigma:\ \forall\ \nu \in \sigma\ ,\ m(\nu) = m(\nu \wedge \mu) + m(\nu \wedge \mu')\} \qquad (2.7)$$

The concept of this notion follows from analogous definition formulated for classical measure theory by Carathe'odory [1]. In [6] it is proved that, for any fuzzy probability measure m on σ, the family σ_m is a soft fuzzy algebra. For presented above families we have :

Theorem 2.1 : For any fuzzy probability measure $m:\sigma \longrightarrow [0,1]$

the following conditions hold:

$$\varepsilon^*_m(\sigma) = \{ \mu \in \sigma:\ \mu \wedge \mu' \in \varepsilon_m(\sigma)\} = \sigma_m \tag{2.8}$$

$$\sigma_m \cap \rho_m(\sigma) = \emptyset \tag{2.9}$$

$$\sigma = (\sigma \cap \rho_m(\sigma)) \cup \sigma_m \tag{2.10}$$

Proof : Let $\mu \in \sigma_m$. In [6] it is shown that then $m(\mu \wedge \mu') = 0$ and $m(\mu \wedge \mu') = 1$. So, $\sigma_m \subset \varepsilon^*_m(\sigma)$.

Conversely, let $\mu \in \varepsilon^*_m(\sigma)$. Then, for any $\eta \in \sigma$ we have (see e.g. [9]):

$$m(\eta) = m(\eta \wedge (\mu \vee \mu')) = m((\eta \wedge \mu) \vee (\eta \wedge \mu')) ,$$

$$0 = m(\eta \wedge (\mu \wedge \mu')) = m((\eta \wedge \mu) \wedge (\eta \wedge \mu')) .$$

Thus, by the valuation property (2.3), we get

$$m(\eta) = m((\eta \wedge \mu) \vee (\eta \wedge \mu')) + m((\eta \wedge \mu) \wedge (\eta \wedge \mu')) =$$

$$= m(\eta \wedge \mu) + m(\eta \wedge \mu') .$$

So, $\varepsilon^*_m(\sigma) \subset \sigma_m$ and the condition (2.8) holds.

Due to (2.8) the condition (2.9) is obvious. Furthermore, we have

$$\sigma = \{\mu \in \sigma:\ \mu \wedge \mu' \notin \varepsilon_m(\sigma)\} \cup \{\mu \in \sigma:\ \mu \wedge \mu' \in \varepsilon_m(\sigma)\} =$$

$$= (\sigma \cap \rho_m(\sigma)) \cup \sigma_m .$$

Theorem 2.2. : For any fuzzy probability measure $m: \sigma \longrightarrow [0,1]$ we have

$$0_X \in \varepsilon_m(\sigma) \tag{2.11}$$

$$(1/2)_X \notin \varepsilon_m(\sigma) \tag{2.12}$$

$$\forall\ \mu,\nu \in \varepsilon_m(\sigma) :\ \mu \vee \nu \in \varepsilon_m(\sigma) \tag{2.13}$$

$$\forall\ \mu \in \varepsilon_m(\sigma), \nu \in \sigma\ :\ \mu \wedge \nu \in \varepsilon_m(\sigma) \tag{2.14}$$

Proof : The condition (2.11) follows from (2.1) and (2.2). Suppose that $(1/2)_X \in \varepsilon_m(\sigma)$. Then we obtain

$m(\mu \vee \nu) = m(\mu) + m(\nu) - m(\mu \wedge \nu) = 0$ and

$m(\mu' \wedge \nu') = m(\mu') + m(\nu') - m(\mu' \vee \nu') = 1$

So, $\varepsilon_m(\sigma)$ is closed under union. Moreover, if $\mu \in \varepsilon_m(\sigma)$ and $\nu \in \sigma$ then $0 \le m(\mu \wedge \nu) \le m(\mu) = 0$ and $1 \ge m(\mu' \vee \nu') \ge m(\mu') = 1$. Therefore, the intersection $\mu \wedge \nu$ belongs to $\varepsilon_m(\sigma)$.

Theorem 2.3. For any fuzzy probability measure $m: \sigma \longrightarrow [0,1]$ we have

$$(1/2)_X \in \rho_m(\sigma) \tag{2.15}$$

$$\rho_m(\sigma) \cap P(X) = \emptyset \tag{2.16}$$

$$\forall\ \mu \in \rho_m(\sigma) :\ \mu' \in \rho_m(\sigma) \tag{2.17}$$

Here P(X) is the family of all crisp subsets of X.

Proof : The condition (2.15) follows from (2.12). The condition (2.16) is obvious because we have $A \cap A' = \emptyset \in \varepsilon_m(\sigma)$ for all

$A \in P(X)$. The definition of $\rho_m(\sigma)$ implies (2.17).

Generalizing the Piasecki's definition of soft fuzzy algebra we propose to accept:

Definition 2.4. : A fuzzy algebra σ will be called a m -soft fuzzy algebra iff it does not contain any m -ill defined element, i.e. $\rho_m(\sigma) \cap \sigma = 0$.

The property (2.10) implies that any m -soft fuzzy algebra contains only such fuzzy random events which are measurable under measure $m: \sigma \longrightarrow [0,1]$. These facts together with some results given in [6] implies the next theorem.

Theorem 2.4. : If a fuzzy probability measure m is defined on m -soft fuzzy algebra σ then it is a fuzzy P -measure on σ.

It is easy to see that the next equalities and inclusions hold:

$$\varepsilon(\sigma_m) \subset \varepsilon_m(\sigma_m) = \varepsilon_m(\sigma) \tag{2.18}$$

$$\rho(\sigma_m) \supset \rho_m(\sigma_m) = \rho_m(\sigma) \tag{2.19}$$

for any fuzzy probability space (X, σ, m).

3. m-ILL DEFINED ELEMENTS: A CASE OF GENERATED FUZZY σ-ALGEBRA

Let $\mathcal{A}$ be a given σ -algebra of crisp subsets from X. In this part we shall consider family of random fuzzy events defined as generated fuzzy σ -algebra $\mathcal{F}(\mathcal{A})$, i.e. family of all $\mathcal{A}$ -measurable fuzzy subsets of X.

The problem of integral representation of a fuzzy probability measure $m: \mathcal{F}(\mathcal{A}) \longrightarrow [0,1]$ was solved by Klement:

Theorem 3.1. [3] : Let $m: \mathcal{F}(\mathcal{A}) \longrightarrow [0,1]$ be a fuzzy probability measure (see [2]) on generated fuzzy σ -algebra $\mathcal{F}(\mathcal{A})$. Then there exists one and only one probability measure $P: \mathcal{A} \longrightarrow [0,1]$ and a P -almost everywhere uniquely determined Markov kernel

$K: (X, \mathcal{A}) \times \mathcal{B}_{[0,1[} \longrightarrow [0,1]$ such that

$$\forall \mu \in \mathcal{F}(\mathcal{A}): m(\mu) = \int_X K(x, [0, \mu(x)[) \, dP(x) . \tag{3.1}$$

Recall that the symbol $\mathcal{B}_{[0,1[}$ denotes the family of all Borel subset from the interval $[0,1[$, and a Markov kernel K is a function fulfilling next two properties:

$$\forall B \in \mathcal{B}_{[0,1[} : K(.,B) : X \longrightarrow [0,1] \text{ is } \mathcal{A} \text{-measurable} \tag{3.2}$$

$\forall\ x \in X : K(x,.) : \mathcal{B}_{[0,1[} \longrightarrow [0,1]$ is a probability measure on $\mathcal{B}_{[0,1[}$. (3.3)

Note that if, for each $x \in X$, the mapping $K(x,.)$ describes the probability measure with uniform distribution on the interval $[0,1[$ then fuzzy probability measure given by (3.1) is a probability of fuzzy events introduced by Zadeh [10].

Come back now to the case of fuzzy probability measure generated by a Markov -kernel K. Then, for the family of all m -ill defined elements connected with generated fuzzy σ -algebra, we get:

<u>Theorem 3.2.</u> : For any fuzzy event $\mu \in \rho_m(\mathcal{F}(\mathcal{A})) \cap \mathcal{F}(\mathcal{A})$ and for any fuzzy event $\nu \in \varepsilon_m(\mathcal{F}(\mathcal{A}))$ we have

$$\Delta\ (\mu,\nu) = \int_{\{x:\mu(x) < \nu(x)\}} K(x,\ [\mu(x),\ \nu(x)[)\ dP(x) = 0 \qquad (3.4)$$

<u>Proof</u> :

$$0 \le \Delta\ (\mu,\nu) \le \int_{\{x:\ \mu(x) < \nu(x)\}} K(x,\ [0,\ \nu(x)[)\ dP(x) \le$$

$$\le \int_X K(x,\ [0,\ \nu(x)[)\ dP(x) = m(\nu) = 0 \qquad \text{q.e.d.}$$

Thus, if for each $x \in X$, the probability measure $K(x,.)$ has the uniform distribution (i.e. m is Zadeh's probability of fuzzy events) then, for each $\mu \in \rho_m(\mathcal{F}(\mathcal{A})) \cap \mathcal{F}(\mathcal{A})$ and $\nu \in \varepsilon_m(\mathcal{F}(\mathcal{A}))$ we get

$$P(\{x:\ \mu(x) < \nu(x)\}) = 0 \qquad (3.5)$$

The last property was used in [5] for definition of the family of all m -ill defined elements from generated fuzzy σ -algebra. Due to the Theorem 3.2 we know that validity of proposed there definition is limited to the case when the mapping $m:\mathcal{F}(\mathcal{A})\longrightarrow [0,1]$ is generated by a Markov -kernel K and a probability measure P such that for a.e. $x \in X$, the distribution $K(x,\ .)$ has a connected support (i.e. its support is an interval or a point). Thus we can say that proposed here Definition 2.4. is a well formalized generalization of ideas given in [5].

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