FUZZY PROBABILITY MEASURES AND THE BAYES FORMULA

Radko MESIAR
Department of Mathematics, Slovak Technical University,
Radlinského 11, 813 68 Bratislava, Czechoslovakia

ABSTRACT

We present a generalization of the Bayes principle for the fuzzy probability measures. We replace the W-disjointness introduced by Piasecki (1985) by the m-disjointess. The validity of the m-Bayes formula for any fuzzy probability measure m is proved.

Keywords: Disjointness, Fuzzy probability measure, Bayes formula

1. Preliminary remarks on fuzzy probability measures

(X, α) will denote a measurable space, i.e. a non-empty set X equipped with a σ -algebra α of crisp subsets of X. A fuzzy sub-Set A will be identified with its membership functionμ, i.e. an α -measurable function μ: $X \rightarrow [0,1]$. For a crisp subset A we will also use $\mu = I_A$. The generated fuzzy σ -algebra of all α -measurable fuzzy subsets of X will be denoted by $F(\alpha)$. Let a fuzzy σ -algebra σ, $\alpha \subset \sigma \subset F(\alpha)$, be given. Klement, Lowen and Schwyhla [1] defined a fuzzy probability measure on σ as a continuous from below mapping $m: \sigma \rightarrow [0,1]$ fulfilling the following conditions :

$$m(0_X) = 0 \ , \ m(1_X) = 1 \qquad (1)$$

$$\forall \ \mu, \nu \in \sigma : \ m(\mu \cup \nu) + m(\mu \cap \nu) = m(\mu) + m(\nu) \qquad (2)$$

Here $\cup$, $\cap$ are the fuzzy connectives of the fuzzy union and the fuzzy intersection. Smets [7] presented a similar definition of fuzzy probability measure, where (1) and (2) hold and the continuity from below is replaced by the continuity from above.

In what follows we will work with original Zadeh's fuzzy connectives of union, intersection and complement, i.e. with max, min and $\mu' = 1 - \mu$.

It is easy to see that Zadeh's concept [8] $m(\mu) = \int_X \mu \, dP$ for a given probability P on (X,α) defines a fuzzy probability measure in the sense of both [1] and [7]. Moreover, in this case we get

$$m(\mu') = 1 - m(\mu) \tag{3}$$

On the other hand, a fuzzy probability measure m on a fuzzy σ-algebra σ induces a probability P on the σ-algebra α,

$$\forall A \in \alpha : \quad P(A) = m(I_A) \tag{4}$$

In what follows, most of the assertions are (or may be) valid in the P-almost-everywhere sense. For the sake of simplicity we will omit " P-a.e. " whenever possible.

The mapping $m(\,.\,/\mu) : \sigma \rightarrow [0,1]$ defined, for any fuzzy probability measure m and for each $\mu \in \sigma$ such that $m(\mu) \neq 0$, by identity

$$m(\nu/\mu) = \frac{m(\nu \cap \mu)}{m(\mu)} \qquad , \ \nu \in \sigma \tag{5}$$

will be called a conditional fuzzy probability given μ (see e.g.[4]).

2. W-Bayes fuzzy partitions and the Bayes formula

Piasecki in [4] has defined, by means of a fuzzy probability measure m, a Bayes fuzzy partition (briefly BFP) of X as the finite or countable sequence $\{\mu_i\}_{i \in I}$ of fuzzy subsets satisfying the following conditions :

(R1) the fuzzy subsets μ_i are pairwise W-separeted, i.e.

$$\mu_i \leq \mu_j' \text{ whenever } i, j \in I , \ i \neq j$$

(R2) $m(\sup\{\mu_i\}) = 1$

(R3) $m(\mu_i) > 0$ for each $i \in I$.

A sequence $\{\mu_i\}_{i \in I}$ satysfying (R1), (R2), (R3) will be called W-BFP.

We recall some of Piasecki's definitions and results.

Definition 2.1. Let σ be a fuzzy σ-algebra (fuzzy algebra). If it does not contain the fuzzy subset $(1/2)_X$, $(1/2)_X \notin \sigma$, then it is called a soft fuzzy σ -algebra (soft fuzzy algebra).

Suppose now that σ is a soft fuzzy σ -algebra. For this case, Piasecki [3] has generalized the definition of classical probability for the fuzzy case as follows.

<u>Definition 2.2.</u> A fuzzy P -measure is a mapping $p : \sigma \longrightarrow [0,1]$ such that

$$\forall\ \mu \in \sigma : \quad p(\mu \cup \mu') = 1 \tag{6}$$

if $\{\mu_i\}_{i\in I}$, $\mu_i \in \sigma$, fulfils (R1) then

$$p(\ \sup_{i\in I}\{\mu_i\}\ ) = \sum_{i\in I} p(\mu_i) \tag{7}$$

If σ is only a fuzzy algebra, we add the assumption $\sup_{i\in I}\{\mu_i\} \in \sigma$ to (7). Any fuzzy P -measure is both Klement's fuzzy probability measure and Smets's one.

We are now interested in the Bayes formula.

<u>Definition 2.3.</u> Let m a be given fuzzy probability measure on a fuzzy σ -algebra σ , and let σ_1 be a fuzzy subalgebra (fuzzy sub - σ -algebra) of σ. Then m satisfies the W -Bayes formula on σ_1 iff for all $\nu \in \sigma$, $m(\nu) > 0$, and any W -BFP $\{\mu_i\}_{i\in I}$, $\mu_i \in \sigma_1$, it holds

$$\forall\ k \in I : \quad m(\mu_k/\nu) = \frac{m(\mu_k)m(\nu/\mu)}{\sum_{i\in I} m(\mu_i)m(\nu/\mu_i)} \tag{8}$$

It is easy to see that (8) is equivalent to

$$m(\nu) = \sum_{i\in I} m(\mu_i)m(\nu/\mu_i) = \sum_{i\in I} m(\nu \cap \mu_i) \tag{9}$$

Note that any fuzzy probability measure m satisfies the W -Bayes formula on α. The main results of Piasecki [4,5] are presented in the following theorem.

<u>Theorem 2.1.</u> Let m be a fuzzy probability measure on a fuzzy σ -algebra σ and let σ_m^W be the smallest fuzzy algebra containing all W -BFP generated by m. Then m satisfies the W -Bayes formula on σ_m^W (i.e. on σ too) iff σ_m^W is soft and m is a fuzzy P -measure on it.

In general, it may happen that σ_m^W is not soft, resp. it is not a fuzzy σ -algebra.

Example 2.1. Let $\sigma = F(\alpha)$, P be a probability on (X,α), $m(\mu) = P(\ \mu > 1/2\ )$. Then $\sigma_m^W = \{\mu \in \sigma,\ P(\mu = 1/2) = 0\}$ is not a fuzzy σ -algebra.

Example 2.2. Let $\sigma = F(\alpha)$, P be a probability on (X,α), $m(\mu) = P(\ \mu > 1/4\ )$. Then $\sigma_m^W = \sigma$ is not soft.

The fuzzy algebra σ_m^W may be defined in two equivalent ways:

$$\sigma_m^W = \{\ \mu \in \sigma,\ \{\mu,\mu'\} \text{ fulfils (R1) and (R2)}\} \tag{10}$$

$$\sigma_m^W = \{\ \mu \in \sigma,\ \exists\ \{\mu_i\} \ni \mu,\ \{\mu_i\} \text{ fulfils (R1) and (R2)}\} \tag{11}$$

Piasecki and Switalski [6] have generalized the notion of the W -disjointess and they have used the following condition to define a Bayes fuzzy partition :

(R1a) the fuzzy subsets μ_i of the sequence $\{\mu_i\}_{i\in I}$ are pairwise F -separated (faintly separated), i.e. $\mu_i \cap \mu_j \le (1/2)_X$ whenever $i,\ j \in I,\ i \ne j$.

A sequence $\{\mu_i\}_{i\in I}$ satisfying (R1a), (R2), (R3) is called F -BFP. Both concepts W - and F - coincide with respect to the P -measures and the Bayes principle. The only exception is in defining the fuzzy algebra σ_m^W . Using (10) or (11) (replacing the W -disjointness by the F -disjointness) we can get two different spaces. If we take m and σ of Example 2.1. ,we get

$$\{\ \mu \in \sigma,\ \exists\ \{\mu_i\} \ni \mu,\ \{\mu_i\} \text{ fulfils (R1a) and (R2)}\} = \sigma$$

$$\{\ \mu \in \sigma,\ \{\mu,\mu'\} \text{ fulfils (R1a) and (R2)}\} = \sigma_m^W\ .$$

Let $A \in \alpha$, $P(A) \in\ ]0,1[$. Then $\mu = I_A + \frac{1}{2} I_{A'} \in \sigma$, but $\mu \notin \sigma_m^W$. Thus, in the F -concept we need to work on the fuzzy algebra $\sigma_m^F = \sigma_m^W$ defined by (10) (for$\{\mu,\mu'\}$are (R1) and (R1a) equivalent) in order to preserve the equivalence of both W - and F -concepts.

3. m -Bayes fuzzy partitions and the Bayes formula

Let m be a given fuzzy probability measure (in sense of Klement) on a fuzzy σ -algebra σ, $\alpha \subset \sigma \subset \mathcal{F}(\alpha)$. In [2] we have proposed to modify the W - and F -disjointness as follows.

Definition 3.1. Elements $\mu, \nu \in \sigma$ will be called m -separated fuzzy subsets iff $m(\mu \cap \nu) = 0$.

Now, we are able to define m -Bayes fuzzy partition.

Definition 3.2. Let $\{\mu_i\}_{i\in I}$ be a sequence of fuzzy subsets of σ satisfying (R2), (R3) and the following condition:

(R1b) the fuzzy subsets μ_i are pairwise m -separated, i.e.

$$m(\mu_i \cap \mu_j) = 0 \quad \text{whenever } i,\ j \in I,\ i \neq j .$$

Then $\{\mu_i\}_{i\in I}$ will be called m -BFP.

Note that if m is a fuzzy P -measure (then σ need to be soft) then any F -BFP is a m -BFP (but not vice versa). Now, we define the fuzzy algebra σ_m^m modifying the expression (10) :

$$\sigma_m^m = \{\ \mu \in \sigma,\ \{\mu, \mu'\} \text{ fulfils (R1b) and (R2) }\} \tag{12}$$

It is easy to see that σ_m^m is a soft fuzzy algebra.

Theorem 3.1. σ_m^m is a soft fuzzy σ -algebra iff there exists $\mu^* \in \sigma$ such that

i) $m(\mu^*) = 1$ and $m(\mu^{*\prime}) = 0$

ii) iff $\nu \in \sigma$, $m(\nu) = 1$ and $m(\nu') = 0$, then $\nu \geq \mu^*$ P-a.e., where $P = m/\alpha$.

Proof. a) Let σ_m^m be a soft fuzzy σ -algebra. Denote

$$\mu^* = \text{ess inf } \{\ \nu \in \sigma,\ m(\nu) = 1 \text{ and } m(\nu') = 0\}$$

(ess inf is taken with respect to P).

Then there exists a non-increasing sequence of fuzzy subsets

$$\{\nu_n\} \longrightarrow \mu^*,\ \nu_n \in \sigma_m^m,\ m(\nu_n) = 1,\ m(\nu_n') = 0 \text{ for } n = 1, 2 \ldots .$$

Then $\mu^* \in \sigma_m^m$. The continuity from below of the measure m leads to $m(\mu^{*\prime}) = \lim_n m(\nu_n') = 0$. As $\mu^* \in \sigma_m^m$, it follows $m(\mu^*) = 1$.

ii) is obvious.

b) Let i) and ii) are fulfilled for a fuzzy subset $\mu^* \in \sigma$. From the definition of σ_m^m (12) we get $\mu \in \sigma_m^m$ iff $m(\mu \cup \mu') = 1$ and $m(\mu \cap \mu') = 0$. The fact $(\mu \cup \mu')' = \mu \cap \mu'$ implies that

$\mu \in \sigma_m^m$ iff $\mu \cup \mu' \geq \mu^*$ P-a.e., i.e. for P-a.e. $x \in X$, we have $\mu(x) \geq \mu^*(x)$ or $\mu(x) \leq \mu^{*\prime}(x)$.

Now, let $\{\mu_n\}$ be a nondecreasing sequence of elements of σ_m^m . Then $\sup_n\{\mu_n\} \in \sigma$. For a fixed $x \in X$, we have either $\mu_n(x) \leq \mu^{*\prime}(x)$ for all $n \in N$, i.e. $\sup_n\{\mu_n(x)\} \leq \mu^{*\prime}(x)$, or there exists $n_0 \in N$, $\mu_{n_0}(x) \geq \mu^*(x)$, i.e. for all $n \geq n_0$, $\mu_n(x) \geq \mu^*(x)$ and $\sup_n\{\mu_n(x)\} \geq \mu^*(x)$, up to a P -null subset of X. Consequently $\sup_n\{\mu_n\} \in \sigma_m^m$.

This fact together with the fact that σ_m^m is fuzzy algebra puts an end to our proof.

Example 3.1. Let m be the Zadeh's fuzzy probability measure $m(\mu) = \int_X \mu \, dP$. Then $\mu^* = 1_X$, $\sigma_m^m = \alpha$ is a σ -algebra and the only Bayes partitions are the crisp ones.

If we take m of Example 2.2. , then $\sigma_m^m = \sigma_m^W$ is not a fuzzy σ -algebra and there exists no μ^* with the properties i) and ii).

By a slight modification of the Definition 2.3. we get the notion of m -Bayes formula. A fuzzy probability measure m satisfies the m -Bayes formula on σ_1 iff (8) is fulfilled for any m -BFP $\{\mu_i\}$.

Theorem 3.2. Let m be a fuzzy probability measure on a fuzzy σ -algebra σ. Then it satisfies the m -Bayes formula on σ.

Proof. Let $\{\mu_i\}_{i\in I}$ be a given m -BFP. It is sufficient to prove the validity of (9) for any $\nu \in \sigma$, i.e.

$$m(\nu) = \sum_{i\in I} m(\nu \cap \mu_i) .$$

We may assume $I = \{1, 2, \dots \}$. Then

$$\nu \cap (\mu_1 \cup \mu_2) = (\nu \cap \mu_1) \cup (\nu \cap \mu_2) \tag{13}$$

$$\nu \cap (\mu_1 \cap \mu_2) = (\nu \cap \mu_1) \cap (\nu \cap \mu_2) \tag{14}$$

The valuation property (2) of m implies

$$m(\nu \cap (\mu_1 \cup \mu_2)) + m(\nu \cap (\mu_1 \cap \mu_2)) = m(\nu \cap \mu_1) + m(\nu \cap \mu_2) \tag{15}$$

μ_1 and μ_2 are m -separated, i.e. $m(\mu_1 \cap \mu_2) = 0$, what implies

$$m(\nu \cap (\mu_1 \cup \mu_2)) = m(\nu \cap \mu_1) + m(\nu \cap \mu_2) \tag{16}$$

By induction it is easy to prove for finite I that

$$m(\nu \cap (\bigcup_I \mu_i)) = \sum_I m(\nu \cap \mu_i) \tag{17}$$

If I is countable, we utilise the continuity from bellow of m to prove (17).

On the other hand, $m(\bigcup_I \mu_i) = m(\nu \cup (\bigcup_I \mu_i)) = 1$ together with the valuation equality

$m(\nu \cup (\bigcup_I \mu_i)) + m(\nu \cap (\bigcup_I \mu_i)) = m(\nu) + m(\bigcup_I \mu_i)$

implies $m(\nu) = m(\nu \cap (\bigcup_I \mu_i))$ q.e.d.

Theorem 3.3. Let m be a fuzzy probability measure on a fuzzy σ -algebra σ. Then m is a fuzzy P -measure on σ^m_m .

Proof. Let $\mu \in \sigma^m_m$. Then $\{\mu, \mu'\}$ fulfills (R2) , i.e. $m(\mu \cup \mu') = 1$, so that (6) is fulfilled.

Let $\{\mu_i\}_{i\in I}$ be a sequence of pairwise W -separated fuzzy subsets of σ^m_m , $\bigcup_I \mu_i \in \sigma^m_m$. Then μ_i are also pairwise m -separated. Now, we can repeat the ideas of the proof of the Theorem 3.2. to get (7) , i.e. $m(\bigcup_I \mu_i) = \sum_I m(\mu_i)$.

It follows that m is a fuzzy P -measure on σ^m_m .

Theorem 3.4. Let m be a fuzzy probability measure on a fuzzy σ -algebra σ. Then σ^m_m is the greatest fuzzy sub -algebra of σ possessing the property that m is a fuzzy P -measure on it.

Proof. It suffices to prove that if m is a fuzzy P -measure on a fuzzy sub -algebra $\sigma_1 \subset \sigma$ then $\sigma_1 \subset \sigma^m_m$, i.e. for every $\mu \in \sigma_1$, $m(\mu \cup \mu') = 1$ and $m(\mu \cap \mu') = 0$. The first property is obvious from (6). (7) that

$$m(\mu \cup \mu') = m(\mu) + m(\mu') \tag{18}$$

From the valuation property (2) we have

$$m(\mu \cup \mu') + m(\mu \cap \mu') = m(\mu) + m(\mu') \tag{19}$$

It follows $m(\mu \cap \mu') = 0$ q.e.d.

Remark. In general $\sigma_m^m \subset \sigma_m^W = \sigma_m^F$. The equality occurs iff the W -Bayes formula holds on σ_m^W . On the other hand, Theorem 3.4. implies that σ_m^m is the greatest fuzzy sub -algebra of σ on which the W -Bayes formula holds. So on σ_m^m all three concepts m - , F - and W - coincide.

Example 3.2. Let m be a symetric fuzzy probability measure on a fuzzy σ -algebra σ (see e.g. [2]), i.e. $\forall\ \mu \in \sigma$:

$m(\mu') = 1 - m(\mu)$. Then $\sigma_m^m = \sigma_m^W$.

References

[1] E.P.Klement, R.Lowen and W.Schwyhla, Fuzzy probability measures,.Fuzzy Sets and Systems **5** (1981) 21-30.

[2] R.Mesiar, Symmetric fuzzy probability measures and the Bayes formula, BUSEFAL **34** (1988) 44-50.

[3] K.Piasecki, Probability of fuzzy events defined as denumerable additivity measure, Fuzzy Sets and Systems **17** (1985) 271-284.

[4] K.Piasecki, On the Bayes formula for fuzzy probability measures, Fuzzy Sets and Systems **18** (1986) 183-185.

[5] K.Piasecki, Note to [4], Fuzzy Sets and Systems **24** (1987) 121-122.

[6] K.Piasecki and Z.Switalski, A remark on the definition of fuzzy P -measure and the Bayes formula, to appear in Fuzzy Sets and Systems (1988).

[7] Ph.Smets, Probability of fuzzy events: an axiomatic approach, Fuzzy Sets and Systems **7** (1982) 153-164.

[8] L.A.Zadeh, Probability measures of fuzzy events, J.Math. Anal.Appl. **23** (1968) 421-427.