MARTINGALE CONVERGENCE THEOREM IN F-QUANTUM SPACES

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In the paper the martingale convergence theorem is proved for sequences of F-observables in F-quantum spaces (see [11]). Let us recall that for compatible observables it was done in [10], for observables in quantum logics in [6],[1] and [7]. The main tool is a representation of F-observables by random variables given by Piasecki [9] and Dvurečenskij [2] and a variant of the Radon-Nikodym theorem (see [4]).

PRELIMINARIES

We recall that a fuzzy measurable space is a couple (Ω, M), where Ω is a nonemty set, and $M \subseteq [0,1]^{\Omega}$ is a soft fuzzy σ-algebra, i.e.

(i) if $1(\omega) = 1$ for any $\omega \in \Omega$, then $1 \in M$;

(ii) $a \in M$ implies $a^{\perp} := 1 - a \in M$;

(iii) $\bigcup_{n \in N} a_n := \sup_{n \in N} a_n \in M$ whenever $(a_n)_{n \in N} \subseteq M$;

(iv) if $1/2(\omega) = 1/2$ for any $\omega \in \Omega$, then $1/2 \notin M$.

The fuzzy meet, $\cap$, is defined, according to Zadeh [12], via $\bigcap_{n \in N} a_n = \inf_{n \in N} a_n$, if $(a_n)_{n \in N} \subseteq M$, and $\bigcap_{n \in N} a_n$ belongs to M. The system M with respect to $\cap$ and $\cup$ is a bounded, distributive , de Morgan σ-lattice with a unary operation $\perp: M \to M$, $a \mapsto a^{\perp}$, satisfying (i) $(a^{\perp})^{\perp} = a$ for any $a \in M$; (ii) if $a \le b$, $a, b \in M$ then $b^{\perp} \le a^{\perp}$.

By an F-observable of (Ω, M) we mean a mapping $x: \mathcal{B}(\mathbf{R}) \to M$ ($\mathcal{B}(\mathbf{R})$ is the Borel σ-algebra of the real line $\mathbf{R}$) such that

(i) $x(E^C) = 1 - x(E)$, $E \in \mathcal{B}(\mathbf{R})$,

(ii) $x(\bigcup_{n \in N} E_n) = \bigcup_{n \in N} x(E_n)$, $(E_n)_{n \in N} \subseteq \mathcal{B}(\mathbf{R})$,

where E^C denotes the set-theoretical complement of E in $\mathbf{R}$.

For example, let $a \in M$ be given, then the mapping $x_a: \mathcal{B}(\mathbf{R}) \to M$

defined via

$$x_a(E) = \begin{cases} a \cap a^{\perp} & \text{if } 0,1 \notin E \\ a^{\perp} & \text{if } 0 \in E,\ 1 \notin E \\ a & \text{if } 0 \notin E,\ 1 \in E \\ a \cup a^{\perp} & \text{if } 0,1 \in E, \end{cases} \qquad (0.1)$$

$E \in \mathcal{B}(\mathbf{R})$, is an F-observable of (Ω, M), called the indicator observable of a fuzzy set $a \in M$.

A P-measure is any mapping $m: M \to [0,1]$ such that $m(a \cup a^{\perp}) = 1$ for any $a \in M$; $m(\bigcup_{n \in \mathbb{N}} a_n) = \sum_{n \in \mathbb{N}} m(a_n)$ whenever $(a_n)_{n \in \mathbb{N}} \subseteq M$, $a_i \le 1 - a_j$, for $i \ne j$.

By a fuzzy probability space we mean any triplet (Ω, M, m), where Ω is a nonvoid set, M is a soft fuzzy σ-algebra and m is a P-measure.

We say that a fuzzy set $a \in M$ is a W-empty set (W-universum) if $a \le 1/2$ $(a \ge 1/2)$. We denote by $W_0(M)$ and $W_1(M)$ the set of all W-empty sets and W-universes, respectively, from M.
Two fuzzy sets a and b from M are W-separated and we write $a \perp b$ if $a \le b^{\perp}$.

According to [9] we define $\mathcal{K}(M)$ as the set of all subsets $A \subset \Omega$ such that there is a fuzzy set $a \in M$ with

$$\{a > 1/2\} \subset A \subset \{a \ge 1/2\}, \qquad (0.2)$$

where $\{a > 1/2\} = \{\omega \in \Omega;\ a(\omega) > 1/2\}$, similarly for $\{a \ge 1/2\}$.
The following result is hold in ([3],[9]):

Theorem 0.1. Let (Ω, M) be a fuzzy measurable space. Then $\mathcal{K}(M)$ is a σ-algebra of subsets of the set Ω. If m is a probability measure on M, then the function $P = P_m$: $\mathcal{K}(M) \to [0,1]$ defined via

$$P_m(A) = m(a),\quad A \in \mathcal{K}(M), \qquad (0.3)$$

where A and a satisfy (0.2), is a P-measure on $\mathcal{K}(M)$ with

$$P_m(\{a = 1/2\}) = 0 \text{ for any } a \in M. \qquad (0.4)$$

Moreover, if m, n are P-measures, $m \ne n$, then $P_m \ne P_n$.
Conversely, let P be any probability measure on $\mathcal{K}(M)$ with (0.4), then the mapping $m_P: M \to [0,1]$ defined via

$$m_P(a) = P(A),\quad a \in M, \qquad (0.5)$$

where a and A fulfil (0.2), is a P-measure on M.
If $P \neq Q$, then $m_P \neq m_Q$. In addition, $m = m_P$ and $P = P_{m_P}$.

In subsequent paragraphs we shall often use the following theorem.

Theorem 0.2. (Representation Theorem (see [2])).
Let x be an F-observable of a fuzzy measurable space (Ω, M). Then there is a $\mathcal{K}(M)$-measurable, real-valued function f on Ω such that

$$\{x(E) > 1/2\} \subset f^{-1}(E) \subset \{x(E) \geq 1/2\} \qquad (0.6)$$

for any $E \in \mathcal{B}(\mathbf{R})$. If g is any $\mathcal{K}(M)$-measurable real-valued function on Ω with (0.6), then

$$\{\omega \in \Omega: \ f(\omega) \neq g(\omega)\} \subset \{x(\emptyset) = 1/2\}. \qquad (0.7)$$

Conversely, let $f: \Omega \to \mathbf{R}$ be any $\mathcal{K}(M)$-measurable function. Then there is an F-observable x with (0.6). If y is any F-observable of (Ω, M) with (0.6), then

$$x(E) \cap y(E^c) \in W_0(M) \qquad (0.8)$$

for any $E \in \mathcal{B}(\mathbf{R})$.

We will write $x \sim f$ if x is an F-observable of (Ω, M) and f is a $\mathcal{K}(M)$-measurable function from Ω into $\mathbf{R}$ such that (0.6) holds.

The sum of any two observables x and y is introduced in [5] as a unique F-observable $x + y$ such that

$$B_{x+y}(t) = \bigcup_{r \in Q} (B_x(r) \cap B_y(t - r)), \qquad (0.9)$$

$t \in \mathbf{R}$, where $B_x(r) = x((-\infty, r))$, $r \in \mathbf{Q}$.

Let h be Borel function then $h \circ x$ is an F-observable of (Ω, M) defined via $h \circ x$: $E \mapsto x(h^{-1}(E))$, $E \in \mathcal{B}(\mathbf{R})$. The product of two F-observables x and y, $x \,.\, y$, is defined as follows

$$x \,.\, y = ((x + y)^2 - x^2 - y^2) / 2. \qquad (0.10)$$

The mean value of an F-observable x in a P-measure m we mean the expression $E(x) := \int_R t \, dm_x(t)$, where $m_x: E \mapsto m(x(E))$, $E \in \mathcal{B}(\mathbf{R})$, is a probability measure on $\mathcal{B}(\mathbf{R})$, if the integral exists and is finite.
We denote by $\mathcal{X}_1(m) = \{x: \mathcal{B}(\mathbf{R}) \to M,\ x$ is an F-observable such that $\int |x| \, dm < \infty \}$.

The indefinite integral, $\int_a x \, \mathrm{d}m$, $a \in M$ is defined via $\int_a x \, \mathrm{d}m :=$ $:= E(x . x_a)$, $a \in M$, where x_a is declared in (0.1).

We will often use the following lemmas (see [2]).

Lemma 0.3. Let $x \sim f$, $y \sim g$ and let h be any Borel function. Then

(i) $x + y \sim f + g$;

(ii) $h \circ x \sim h \circ f$;

(iii) $x . y \sim f . g$;

(iv) if $f \geq 0$, then $x([0,\infty)) = x(\mathbb{R})$.

Lemma 0.4. Let $x \sim f$, m be a P-measure on M and let P_m be a probability measure on $\mathcal{K}(M)$ defined via (0.3). Let $a \in M$ and $A \in \mathcal{K}(M)$ be related through (0.2). Then

(i) $m(x(E)) = P_m(f^{-1}(E))$, $E \in \mathcal{B}(\mathbb{R})$;

(ii) $x_a \sim I_A$, where I_A is the indicator of A;

(iii) $\int_a x \, \mathrm{d}m = \int_a f \, \mathrm{d}P_m$.

Let us remark, that the mapping $n\colon M \to \mathbb{R}$ such that $n(a) = \int_a x \, \mathrm{d}m$ is the signed measure on M, and $m(a) = 0$ implies $n(a) = 0$. Indeed, $n(a) = \int_a x \, \mathrm{d}m = \int_A f \, \mathrm{d}P_m = 0$ because $P_m(A) = m(a) = 0$.

CONDITIONAL EXPECTATION

Troughout the paper we shall assume that (Ω, M) be a fuzzy measurable space, where Ω is a nonempty set and M is a soft fuzzy σ-algebra.

Definition 1.1. Let x, y be F-observables. We will say that x, y are equal almost everywhere with respect to a P-measure m ($x = y$ a.e.$[m]$) if $m((x - y)(\{0\})) = 1$.

We will say, that x is less or equal to y almost everywhere with respect to a P-measure m ($x \leq y$ a.e.$[m]$) if

$m((y - x)([0,\infty))) = 1$.

Lemma 1.2. Let $x \sim f$, $y \sim g$. Then

1. $x = y$ a.e.$[m]$ if and only if $f = g$ a.e.$[P_m]$;
2. $x \geq y$ a.e.$[m]$ if and only if $f \geq g$ a.e.$[P_m]$.

Proof. 1. Because $x \sim f$, $y \sim g$ implies $x - y \sim f - g$ (Lemma 0.3), i.e. $\{(x-y)(E) > 1/2\} \subset (f-g)^{-1}(E) \subset \{(x-y)(E) \geq 1/2\}$, for every $E \in \mathcal{B}(\mathbb{R})$, the equality

$$m((x-y)(E)) = P_m((f-g)^{-1}(E)),$$

for every $E \in \mathcal{B}(\mathbb{R})$ is fulfiled (Th.0.1). Since

$$P_m((f-g)^{-1}(\{0\})) = 1 - P_m((f-g)^{-1}(\mathbb{R} - \{0\}))$$

the previous assertion 1 is proved.

Indeed $x = y$ a.e.$[m]$ is equivalent with the equality

$$m(x-y)(\{0\}) = 1$$

and equality

$$P_m((f-g)^{-1}(\mathbb{R} - \{0\})) = P_m(\{\omega \in \Omega: f(\omega) \neq g(\omega))\}) = 0$$

is equivalent with the equality $f = g$ a.e.$[P_m]$.

2. The equality

$$m((x-y)(E)) = P_m((f-g)^{-1}(E))$$

for every $E \in \mathcal{B}(\mathbb{R})$ implies

$$m((x-y)([0,\infty))) = P_m((f-g)^{-1}([0,\infty))),$$

and

$$1 = m((x-y)([0,\infty))) = P_m((f-g)^{-1}([0,\infty)))$$

is equivalent with $x \geq y$ a.e.$[m]$ and $f \geq g$ a.e.$[P_m]$ respectively.

Q.E.D.

Lemma 1.3. Let $M_0 \subset M$ be a fuzzy soft subσ-algebra. Let x be an integrable F-observable on M. Then there exists an F-observable y on M_0 such that $\int_a y\,dm = \int_a x\,dm$ for every $a \in M_0$.

If $\int_a x\,dm = \int_a z\,dm$ for another an F-obsrvable z on M_0, then

$$y = z \text{ a.e.}[m/M_0].$$

Proof. Put $n(a) = \int_a x\,dm$ for every $a \in M_0$. The mapping n is a signed measure on M_0 and $m(a) = 0$ implies $n(a) = 0$. According to the Radon - Nikodym theorem (see [4]) then there exists an

F-observable y on M_0, such that $n(a) = \int_a y\,dm$ for every $a \in M_0$.

If $n(a) = \int_a z\,dm$ for another an F-observable z, then

$y = z$ a.e. $[m/M_0]$.

Q.E.D.

Definition 1.4 Let x be an integrable F-observable on M, $M_0 \subset M$ be a fuzzy soft subσ-algebra. Then by a version of conditional expectation of the F-observable x for M_0 ($E(x/M_0)$) we understand any F-observable y on M_0 with the property

$\int_a y\,dm = \int_a x\,dm$ for every $a \in M_0$.

Especially, if z is an F-observable on M, then $z(\mathcal{B}(\mathbb{R}))$ is a sub-σ-algebra and $y = E(x/z(\mathcal{B}(\mathbb{R})))$ is called a version of conditional expectation of the F-observable x with respect to the F-observable z. It is denoted by $E(x/z)$.

MARTINGALS AND SUBMARTINGALS

In this part we introduce the definition of submartingals and martingals.

Definition 2.1. Let $(\Omega, M.m)$ be an F-probability space, $(M_n)_{n \in \mathbb{N}}$ be a sequence of fuzzy soft σ-algebras of M, $(x_n)_{n \in \mathbb{N}}$ be a sequence of F-observables. Then the sequence $((x_n, M_n))_{n \in \mathbb{N}}$ will be called a submartingal if it holds:

1. $M_n \subset M_{n+1} \subset M$ for all $n = 1, 2, \ldots$;
2. x_n be the F-observable on the F-measurable space (Ω, M), such that $x_n(E) \in M_n$ for every $E \in \mathcal{B}(\mathbb{R})$, $n = 1, 2, \ldots$;
3. $x_n \le E(x_{n+1}/M_n)$ a.e. $[m]$, $n = 1, 2, \ldots$.

A submartingal will be called a martingal, if

4. $x_n = E(x_{n+1}/M_n)$ a.e. $[m]$, $n = 1, 2, \ldots$.

Lemma 2.2. Let $((x_n, M_n))_{n \in \mathbb{N}}$ be a submartingal (martingal) on F-probability space (Ω, M, m). Then $((f_n, \mathcal{K}(M_n)))_{n \in \mathbb{N}}$, where $x_n \sim f_n$, $n = 1, 2, \ldots$, is a submartingal (martingal) on the probability space $(\Omega, \mathcal{K}(M), P_m)$.

Proof. 1. According to the definition of $\mathcal{K}(M)$ the following is

fulfiled: if M_1, M_2 are soft fuzzy σ-algebras of M, $M_1 \subset M_2 \subset M$, then $\mathcal{K}(M_1) \subset \mathcal{K}(M_2) \subset \mathcal{K}(M)$.
Indeed, if $A \in \mathcal{K}(M_1)$ then there is a fuzzy set $a \in M_1$ and also $a \in M_2$ such that (0.2) is hold. Therefore $A \in \mathcal{K}(M_2)$.

2. For every F-observable x on (Ω, M) there is an $\mathcal{K}(M)$-measurable, real-valued function f on Ω, $x \sim f$ (see Th. 0.2).

3. If $x_n \le E(x_{n+1}/M_n)$ a.e.$[m]$, $n = 1, 2, \dots$, f_n, g_n are the $\mathcal{K}(M)$ measurable, real-valued functions, $x_n \sim f_n$, $E(x_{n+1}/M_n) \sim g_n$, then $f_n \le g_n$ a.e.$[P_m]$ (see Lemma 1.2).

Q.E.D.

Definition 2.3. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ of F-observables on (Ω, M) converges to an F-observable x on (Ω, M) almost everywhere in an F-state m, if

$$m\Big(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} (x - x_n)(-\varepsilon, \varepsilon)\Big) = 1, \text{ for every } \varepsilon > 0.$$

Lemma 2.4. Let x, x_n be the observables on (Ω, M), let f_n, f be $\mathcal{K}(M)$-measurable functions on Ω, such that $x \sim f$, $x_n \sim f_n$, for any $n \ge 1$. A sequence $(x_n)_{n\in\mathbb{N}}$ converges to x almost everywhere in an F-state m, if and only if a sequence $(f_n)_{n\in\mathbb{N}}$ converges to f almost everywhere in the measure P_m.

Proof. We remark that the following assertion is fulfiled.
If $(a_n)_{n\in\mathbb{N}}$ is a sequence from M, then

$$\{\bigcup_{n\in\mathbb{N}} a_n > 1/2\} = \bigcup_{n\in\mathbb{N}} \{a_n > 1/2\}, \quad \{\bigcup_{n\in\mathbb{N}} a_n \ge 1/2\} \supset \bigcup_{n\in\mathbb{N}} \{a_n \ge 1/2\},$$

$$\{\bigcap_{n\in\mathbb{N}} a_n > 1/2\} \subset \bigcap_{n\in\mathbb{N}} \{a_n > 1/2\}, \quad \{\bigcap_{n\in\mathbb{N}} a_n \ge 1/2\} = \bigcap_{n\in\mathbb{N}} \{a_n \ge 1/2\}.$$

Using Lemma 0.3 we have $x - x_n \sim f - f_n$, i.e.

$$\{(x - x_n)(E) > 1/2\} \subseteq (f - f_n)^{-1}(E) \subseteq \{(x - x_n)(E) \ge 1/2\}$$

for every $E \in \mathcal{B}(\mathbb{R})$, $n = 1, 2, \dots$.
By the previous assertion

$$\{\bigcap_{n=k}^{\infty} (x - x_n)(E) > 1/2\} \subseteq \bigcap_{n=k}^{\infty} \{(x - x_n)(E) > 1/2\} \subseteq$$

$$\subseteq \bigcap_{n=k}^{\infty} (f - f_n)^{-1}(E) \subseteq \bigcap_{n=k}^{\infty} \{(x - x_n)(E) \ge 1/2\} =$$

$$= \{\bigcap_{n=k}^{\infty} (x - x_n)(E) \ge 1/2\},$$

and so

$$\{\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}(x-x_n)(E)>1/2\}=\bigcup_{k=1}^{\infty}\{\bigcap_{n=k}^{\infty}(x-x_n)(E)>1/2\}\subset$$

$$\subset\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}(f-f_n)^{-1}(E)\subset\bigcup_{k=1}^{\infty}\{\bigcap_{n=k}^{\infty}(x-x_n)(E)\geq 1/2\}\subset$$

$$\subset\{\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}(x-x_n)(E)\geq 1/2\}$$

for every $E\in\mathcal{B}(\mathbb{R})$.

By the definition of P_m

$$m(\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}(x-x_n)(-\varepsilon,\varepsilon))=P_m(\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}(f-f_n)^{-1}([-\varepsilon,\varepsilon]))$$

for every $\varepsilon>0$.

Q.E.D.

Theorem 2.5. Let $((x_n,M_n))_{n\in\mathbb{N}}$ be a submartingal, $\sup_{n\in\mathbb{N}}E(|x_n|)<$ $<\infty$, then there is an F-observable $x\colon\mathcal{B}(\mathbb{R})\to M_0$, where M_0 is the smallest fuzzy soft σ-algebra containing the union $\bigcup_{n=1}^{\infty}M_n$, and x_n converges to x a.e.$[m]$.

Proof. From Lemma 2.1, the sequence $((f_n,\mathcal{K}(M_n)))_{n\in\mathbb{N}}$ is a submartingal on the probability space $(\Omega,\mathcal{K}(M),P_m)$, $x_n\sim f_n$, $n\in\mathbb{N}$.

The condition $\sup_{n\in\mathbb{N}}E(|x_n|)<\infty$ implies $\sup_{n\in\mathbb{N}}E(|f_n|)<\infty$. Then correspoding with the classical theory, for example see [8], then there is a S_0-measurable real-valued function f ($S_0=\sigma(\bigcup_{n=1}^{\infty}\mathcal{K}(M_n))$ is the smallest σ-algebra containing the union $\bigcup_{n\in\mathbb{N}}M_n$), such that f_n converges to f a.e.$[P_m]$. We note, that the function f is an S_0-measurable iff $f^{-1}(E)\in S_0$ for every $E\in\mathcal{B}(\mathbb{R})$.

Let now $A\in\bigcup_{n\in\mathbb{N}}\mathcal{K}(M_n)$. Then there is $k\in\mathbb{N}$, such that $A\in\mathcal{K}(M_k)$, and so then there is $a\in M_k$, such that

$\{a>1/2\}\subset A\subset\{a\geq 1/2\}$ $(a\sim A)$.

Therefore we have: for every $A\in\bigcup_{n\in\mathbb{N}}\mathcal{K}(M_n)$ there is $a\in M_0$ such that $a\sim A$.

Because $\mathcal{K}(M_0)=\{A\subset\Omega;$ such that there is $a\in M_0\colon a\sim A\}$, is $\mathcal{K}(M_0)\supset\bigcup_{n\in\mathbb{N}}\mathcal{K}(M_n)$, too.

So, the function f is $\mathcal{K}(M_0)$-measurable and by Th. 0.2 there is an F-observable x, $x\colon\mathcal{B}(\mathbb{R})\to M_0$, such that $x\sim f$. By Lemma 2.4 x_n convergences to x a.e.$[m]$.

Q.E.D.

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