ON A REPRESENTATION THEOREM OF OBSERVABLES IN ORDERED SPACES

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ABSTRACT. We present a generalization of a representation lemma for σ-observables, known for quantum logics, to a weakly orthocomplemented σ-poset. As a special case we also obtain a representation theorem for F-quantum spaces.

1. INTRODUCTION

If P is a quantum logic, $x, y : \mathcal{B}(R) \to P$ are two observables and $x(\mathcal{B}(R)) \subset y(\mathcal{B}(R))$, then there is a Borel measurable function $T : R \to R$ such that $x = y \circ T^{-1}$ (see e.g. [2], [11]). In this note, we present a generalization of this lemma. As a special case we obtain also a representation lemma in F-quantum spaces ([3,8,9]). This result enables us to prove a variant of the ergodic theorem in F-quantum spaces ([5]) and probably some other limit theorems, too ([4]).

We shall say that a partially ordered set P with a mapping $a \to a'$ is a weakly orthocomplemented σ-poset, if (i) $(a')' \geq a$ for every $a \in P$; (ii) if $a, b \in P$, $a \leq b$, then $b' \leq a'$; (iii) if $(a_i)_i \subset P$, $a_i \leq a'_j$ $(i \neq j)$, then there exists $\bigvee_i a_i$ in P; (iv) $a \neq a'$ for every $a \in P$. These posets were studied e.g. in [1].

We note that for every $a \in P$ we have $a' = a'''$. Indeed, since $a \leq a''$, then $a' \geq a'''$ and $a' \leq (a')''$. Analogically, we may show that $\{a \in P : \ a = a''\} = \{b' : \ b \in P\}$.

Two elements a and b from P are orthogonal and we write $a \perp b$ if $a \leq b'$.

A set F of functions $f : \ X \to [0,1]$ is an F-quantum space, if the following conditions are satisfied: $a)$ F contains the constant function 0 and does not contain the constant function 1/2; $b)$ if $f \in F$, then $f' = 1 - f \in F$; $c)$ if $f_n \in F$ $(n = 1, 2, ...)$, then $\sup_n f_n \in F$.
It is clear that every F-quantum space satisfies the above assumptions $(i) - (iv)$.

Motivated by some physical reasons, J. Pykacz ([7]) suggested to substitute the property $c)$ in F-quantum spaces by a weaker one: $c_1)$ if $f_n \in F$ $(n = 1, 2, ...)$ and $f_n \leq f'_m = 1 - f_m$ $(n \neq m)$, then $\sup_n f_n \in F$. Evidently, also the weaker form of an F-quantum space satisfies the above assumptions. It is simple to show that it is not true , that $f \vee f' = 1$, in general.

A q-σ-algebra Q ([10]) is a family of subsets of a given set X satisfying the following conditions: 1) $\emptyset \in Q$; 2) if $A \in Q$ then $X \setminus A \in Q$; 3) if $A_n \in Q$ $(n = 1, 2, ...)$ and A_n are pairwise disjoint, then $\bigcup_n A_n \in Q$.
Put now $F = \{\chi_A : \ A \in Q\}$. F satisfies the assumptions $a)$, $b)$, $c_1)$, and, hence, the assumptions $(i) - (iv)$, too.

Let H be a pre-Hilbert space, P be the set of all closed subspaces of H. Then P satisfies the assumptions $(i)-(iv)$ (with $A' = \{x \in H;\ (x,a) = 0 \text{ for any } a \in A\}$), but P need not be a logic.

There are examples of subspaces of H such that $A'' \neq A$, and $A \vee A' \neq H$. Let, in $P := \{0, a, a', b, b', b'', c, d, 1\}$,the partially ordering be given according to Fig. 1. The orthocomplementation $a \to a'$ in P is defined by the following relations: $c' = 1$, $0' = 1$, $1' = c$, $d' = c$.

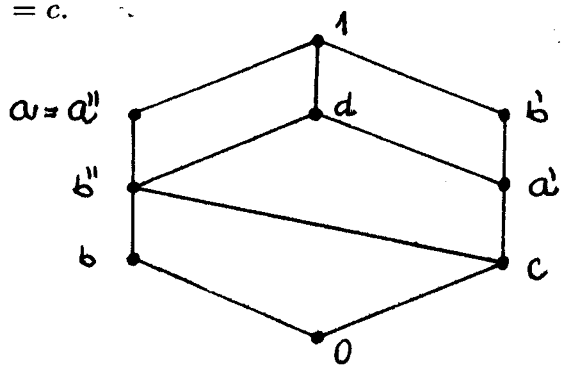

FIG. 1.

2. REPRESENTATION OF OBSERVABLES

Definition 1. Let $\mathcal{B}$ denote a σ-algebra of subsets of a nonvoid set Y. Let P be a weakly orthocomplemented σ-poset. A mapping $x:\ \mathcal{B} \to P$ is called a σ-homomorphism if

1) $x(E^c) = (x(E))'$ for every $E \in \mathcal{B}$;
2) $x(E) \perp x(F)$ if $E, F \in \mathcal{B}$, $E \cap F = \emptyset$;
3) if $E_n \in \mathcal{B}$ $(n = 1, 2, \ldots)$ and $E_i \cap E_j = \emptyset$ for $i \neq j$, then $x(\bigcup_n E_n) = \bigvee_n x(E_n)$.

In particular, if $\mathcal{B} = \mathcal{B}(R)$ ($\mathcal{B}(R)$ is the set of all Borel subsets in R), then σ-homomorphism x is called an observable.

It is not difficult to see, that every σ-homomorphism satisfies the following conditions: If A_n $(n = 1, 2, \ldots)$ are subsets from $\mathcal{B}$ and x is an σ-homomorphism , then $\bigvee_n x(A_n)$ and $\bigwedge_n x(A_n)$ exist in P, and

$$x(\bigcup_n A_n) = \bigvee_n x(A_n), \quad x(\bigcap_n A_n) = \bigwedge_n x(A_n).$$

If $A_1 \subset A_2$, then $x(A_1) \leq x(A_2)$.

Theorem 1. *Let P be a weakly orthocomplemented σ-poset. Let $y, z:$ $\mathcal{B}(R) \to P$ be two observables and $z(\mathcal{B}(R)) \subseteq y(\mathcal{B}(R))$. Then there is a Borel measurable mapping $T:\ R \to R$, such that $z(E) = y(T^{-1}(E))$ for every $E \in \mathcal{B}(R)$.*

Proof. First we prove the following lemma:
If $A, B, C \in \mathcal{B}(R)$, $y(A) = z((-\infty, r))$, $y(B) = z((-\infty, s))$, $y(C) = z((-\infty, t))$, $A \subset C$ and $r \leq s \leq t$, then there is a $D \in \mathcal{B}(R)$ such that $A \subset D \subset C$ and $y(D) = z((-\infty, s))$.

Indeed, it suffices to put $D = (A \cup B) \cap C$. Then

$$\begin{aligned} y(D) &= (y(A) \vee y(B)) \wedge y(C) \\ &= \Big(z((-\infty, r)) \vee z((-\infty, s))\Big) \wedge z((-\infty, t)) \\ &= z\Big(((-\infty, r) \vee (-\infty, s)) \wedge (-\infty, t)\Big) \\ &= z((-\infty, s)). \end{aligned}$$

Now let $(r_i)_i$ be a sequence of all rational numbers. First we shall construct a sequence $(E_i)_i$ of Borel sets such that $r_i < r_j$ implies $E_i \subset E_j$, $y(E_i) = z((-\infty, r_i))$. By the assumption, there are $F_i \in \mathcal{B}(R)$ such that $y(F_i) = z((-\infty, r_i))$. We put $E_1 = F_1$ and define $(E_n)_n$ by the induction:

1. If $r_n > r_k$, $r_k = \max\{r_1, ..., r_{n-1}\}$, then $E_n = E_k \cup F_n$.
2. If $r_n < r_i$, $r_i = \min\{r_1, ..., r_{n-1}\}$, then we put $E_n = E_i \cap F_n$.
3. If there are $i, k \in \{1, ..., n-1\}$ such that $r_i = \min\{r_m : r_m > r_n, m = 1, ..., n-1\} > r_k = \max\{r_m : r_m < r_n, m = 1, ..., n-1\}$, then we use the previous lemma.

If now we put $G_i = E_i \setminus \bigcap_{j=1}^{\infty} E_j$, then evidently $\bigcap_{i=1}^{\infty} G_i = \emptyset$ and $r_i < r_j$ yields $G_i \subset G_j$. Moreover,

$$\begin{aligned} y(G_i) &= y\Big(E_i \cap (\bigcap_j E_j)^c\Big) = y(E_i) \wedge \Big(\bigwedge_j y(E_j)\Big)' \\ &= z((-\infty, r_i)) \wedge \Big(\bigwedge_j z((-\infty, r_j))\Big)' = z\Big((-\infty, r_i) \cap (\bigcap_j (-\infty, r_j))^c\Big) \\ &= z((-\infty, r_i)) \end{aligned}$$

Now we define for every $t \in R$

$$T(t) = \begin{cases} \inf\{r_i;\ t \in G_i\}, & \text{if } t \in \bigcup_{j=1}^{\infty} G_j \\ 0 & \text{otherwise.} \end{cases}$$

Then $T: R \to R$ is a well-defined mapping and

$$T^{-1}((-\infty, r_i)) = \bigcup\{G_j :\ r_j < r_i\}, \text{ if } r_i \le 0,$$

$$T^{-1}((-\infty, r_i)) = \bigcup\{G_j :\ r_j < r_i\} \cup (\bigcup_k G_k)^c, \text{ if } r_i > 0.$$

We see, that T is Borel measurable. Moreover,

$$\begin{aligned} y\Big(T^{-1}((-\infty, r_i))\Big) &= \bigvee\{y(G_j) :\ r_j < r_i\} = \\ &= \bigvee\{z((-\infty, r_j)) :\ r_j < r_i\} = z((-\infty, r_i)) \end{aligned}$$

if $r_i \le 0$, and similarly as in the second case we have

$$y\Big(T^{-1}((-\infty, r_i))\Big) = z((-\infty, r_i)) \text{ for every } r_i.$$

Since, $K := \{E \in \mathcal{B}(R);\ y(T^{-1}(E)) = z(E)\}$ includes $C := \{(-\infty, r);\ r \in Q\}$, and y, z and T^{-1} are σ-homomorphisms, we see that K is a σ-algebra. Therefore, $\mathcal{B}(R) = \sigma(C) \subset K$, so that $y(T^{-1}(E)) = z(E)$ for every $E \in \mathcal{B}(R)$.

□

Corollary 1. *If $\mathcal{B} = \mathcal{B}(Y)$ is a Borel σ-algebra of a complete separable metric space Y, then the statement of Theorem 1 is valid for all observables z, y; $\mathcal{B} \to P$ such that $z(\mathcal{B}) \subset y(\mathcal{B})$.*

Proof. Due to a classical theorem of the separable descriptive theory ([6, par. 33, Th. 2]), we have that $\mathcal{B}(Y)$ is σ-isomorphic to $\mathcal{B}(R)$.

□

Corollary 2. *Let L be a quantum logic, x be an observable, $\tau : L \to L$ be an x-measurable σ-homomorphism (i.e. $\tau\Big(x\big(\mathcal{B}(R)\big)\Big) \subset x\big(\mathcal{B}(R)\big)$). Then there exists a Borel measurable mapping $T: R \to R$ such that $\tau(x(E)) = x(T^{-1}(E))$ for every $E \in \mathcal{B}(R)$.*

Proof. Put $z = \tau \circ x$, $y = x$.

□

Now we shall present a theorem which is in certain sence a generalization of Theorem 1. It holds, in particular, in more general topological spaces; of course, the observables y and z are assumed to satisfy some further conditions.

Theorem 2. *Let $\mathcal{B}$ be a σ-algebra of subsets of a set $Y \neq \emptyset$ containing a countable generator of $\mathcal{B}$. Let P be a weakly orthocomplemented σ-poset. Let $y, z : \mathcal{B} \to P$ be σ-homomorphisms such that $y(E) = y(\emptyset)$ iff $E = \emptyset$, and $z(\mathcal{B}) \leq y(\mathcal{B})$. Then there is a $\mathcal{B}$-measurable mapping $T: Y \to Y$ such that $z = y \circ T^{-1}$.*

Proof. Let $(F_i)_{i=1}^{\infty}$ be a countable generator of $\mathcal{B}$. Without loss of generality we may assume that $\bigcup_{i=1}^{\infty} F_i = Y$; in the opposite case $(F_i)_{i=0}^{\infty}$ is also a generator of $\mathcal{B}$, where $F_0 = (\bigcup_{i=1}^{\infty} F_i)^c$.
For any $t \in Y$, we define

$$F_t = \{y(E) : t \in E \in \mathcal{B}\} \text{ and } G_t = \{G \in \mathcal{B} : z(G) \in F_t\}.$$

Due to the injectivity of y, G_t is a maximal σ-filter of $\mathcal{B}$, that is (1) $G_t \neq \emptyset$; (2) $G_n \in G_t$, $n \geq 1$, implies $\bigcap_n G_n \in G_t$; (3) $G \subset H \in \mathcal{B}$, $G \in G_t$, then $H \in G_t$; (4) G_t contains exactly one of the elements A, A^c for every $A \in \mathcal{B}$.
Define a sequence $\big(F_i(t)\big)_{i=1}^{\infty}$ via

$$F_i(t) = \begin{cases} F_i, & \text{if } F_i \in G_t, \\ F_i^c, & \text{if } F_i \notin G_t. \end{cases}$$

Then $F_i(t) \in G_t$ for any $i \geq 1$, and the intersection $C = \bigcap_{i=1}^{\infty} F_i(t)$ is non-void element of G_t. Indeed, in the opposite case we would have $\emptyset = C \in G_t$, consequently, $G_t = \mathcal{B}$. Therefore, there exists some point $T(t) \in Y$, say, such that $T(t) \in C$.
We claim to show that the mapping $T : Y \to Y$ defined via $t \to T(t)$, $t \in Y$, is measurable and $y(T^{-1}(G)) = z(G)$ for any $G \in \mathcal{B}$.
Due to our assumptions, y is injective. Hence, for any $G \in \mathcal{B}$, there exists a unique $E \in \mathcal{B}$ such that $z(G) = y(E)$. We assert that $T^{-1}(G) = E$.
Let $t \in T^{-1}(G)$, then $T(t) \in G$, and suppose $t \notin E$, then $t \in E^c$, and $z(G^c) =$

$(z(G))' = (y(E))' = y(E^c)$, i.e., $E^c \in F_t$ and $G^c \in G_t$.
Since the system $X = \{A \in \mathcal{B} : A \cap C = \emptyset \text{ or } C \subset A\}$ is a σ-algebra containing all $F_i(t)$ $(i \geq 1)$, i.e. $X = \mathcal{B}$, $C \subset A$ for every $A \in G_t$ which entails $T(t) \in G^c$ and this contradicts $t \notin E$.

If now $t \in E$, then $y(E) \in F_t$. Because $y(E) = z(G)$, we have $G \in G_t$, $T(t) \in G$, i.e. $t \in T^{-1}(G)$.
We have proved that $T^{-1}(G) = E$ and $z(G) = y(E) = y(T^{-1}(G))$.

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